Second-Order Linear ODEs

A second order ODE is called **linear** if it can be written as

$$y'' + p(t)y' + q(t)y = r(t). \tag{0.1}$$

It is called **homogeneous** if $r(t) = 0$, and **nonhomogeneous** otherwise. We shall assume the following important theorem about linear ODEs without proof.

Theorem 1 (Existence and Uniqueness Theorem). *Let p(t), q(t) and r(t) be continuous functions on some open interval I. Let $t_0 \in I$. Then for any numbers y_0 and y_0', equation* (0.1) *with initial conditions $y(t_0) = y_0, y'(t_0) = y_0'$ has a unique solution on the interval I.*

Proof. Beyond the scope of this course. □

Remark 1. *The above theorem can be extended to higher order linear ODEs as well. Write the corresponding IVP and the existence-uniqueness theorem yourself.*

1 Homogeneous Second-Order Linear ODEs

Consider a general homogeneous second-order linear ODE of the form

$$y'' + p(t)y' + q(t)y = 0. \tag{1.1}$$

Theorem 2. *The set of all solutions of a homogeneous second-order linear ODE is a vector space.*

Proof. It is easy to verify that if y_1 and y_2 are any two solutions of (1.1) then $c_1y_1 + c_2y_2$ is also a solution for any constants c_1 and c_2. □

Remark 2. *Note that the above theorem is not valid for nonlinear ODEs or nonhomogeneous linear ODEs.*

Question: What is the dimension of the solution space of homogeneous second-order linear ODE?

Recall the definition of linear dependence and independence. First we claim that if $p(t)$ and $q(t)$ are continuous on an open interval I then there are at least two solutions of (1.1) which are linearly independent on I. To prove the claim let us take some $t_0 \in I$, and let $y_1(t)$ and $y_2(t)$ be the unique solutions of (1.1) satisfying the initial conditions $y_1(t_0) = 1, y_1'(t_0) = 0$ and $y_2(t_0) = 0, y_2'(t_0) = 1$, respectively. (Note that such $y_1(t)$ and $y_2(t)$ exist by Theorem 1.) It is easy to see in this case that $y_1(t)$ and $y_2(t)$ are linearly independent on I. Hence the dimension of the solution space is at least two.

We would show that the dimension is, in fact, equal to two. Let $y(t)$ be any solution of (1.1). Consider $Y(t) = y(t_0)\ y_1(t) + y'(t_0)\ y_2(t)$, where $y_1(t)$ and $y_2(t)$ are as defined above. Since $Y(t)$ is a linear combination of solutions $y_1(t)$ and $y_2(t)$, $Y(t)$ is also a solution. Furthermore, $Y(t_0) = y(t_0)$ and $Y'(t_0) = y'(t_0)$. By the uniqueness, $Y(t) = y(t)$ for all $t \in I$, and hence $y(t)$ is a linear combination of $y_1(t)$ and $y_2(t)$. This proves that the dimension is equal to two. Thus we have obtained the following theorem.

Theorem 3. *If $p(t)$ and $q(t)$ are continuous on an open interval I, then the solution space of $y'' + p(t)y' + q(t)y = 0$ is two-dimensional.*

Exercise: Generalize the previous theorem to n-th order homogeneous linear ODEs with continuous coefficients.

Remark 3. *Note that the theorem is not true in general without the continuity assumptions.*

Summary: In order to find all solutions of a homogeneous second-order linear ODE of the form (1.1) with continuous coefficients it is enough to find any pair of linearly independent solutions.

Question: How to check whether two functions are linearly independent or not?

Suppose that $f_1(t), f_2(t), \ldots, f_n(t)$ are functions defined on an open interval I. Assume that $f_1(t), f_2(t), \ldots, f_n(t)$ are linearly dependent. Then there exist constants $c_1, c_2, \ldots, c_n$ not all zeros, such that $c_1 f_1(t) + c_2 f_2(t) + \cdots + c_n f_n(t) = 0$ for all $t \in I$. If we assume that $f_1, f_2, \ldots, f_n$ are $(n-1)$ times differentiable on I, then we have

$$\begin{array}{ccccccccc}
c_1 f_1(t) & + & c_2 f_2(t) & + & \cdots & + & c_n f_n(t) & = & 0 \\
c_1 f_1'(t) & + & c_2 f_2'(t) & + & \cdots & + & c_n f_n'(t) & = & 0 \\
 & & & & \vdots & & & & \\
c_1 f_1^{(n-1)}(t) & + & c_2 f_2^{(n-1)}(t) & + & \cdots & + & c_n f_n^{(n-1)}(t) & = & 0
\end{array}$$

for all $t \in I$. Since the above homogeneous system has a nonzero solution $(c_1, c_2, \dots, c_n)$, the determinant of the coefficient matrix must be zero for every $t \in I$. This motivates the following definition of the Wronskian function.

Definition 1 (Wronskian). *For n real-valued functions $f_1, f_2, \dots, f_n$, which are $(n-1)$ times differentiable on an open interval I, the Wronskian $W(f_1, f_2, \dots, f_n)$ as a function on I is defined by*

$$W(f_1, f_2, \dots, f_n)(t) = \begin{vmatrix} f_1(t) & f_2(t) & \cdots & f_n(t) \\ f_1'(t) & f_2'(t) & \cdots & f_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(t) & f_2^{(n-1)}(t) & \cdots & f_n^{(n-1)}(t) \end{vmatrix}, \quad t \in I. \tag{1.2}$$

Now we have the following theorem.

Theorem 4. *If $(n-1)$ times differentiable functions $f_1, f_2, \dots, f_n$ on an open interval I are linearly dependent on I, then $W(f_1, f_2, \dots, f_n)(t) = 0$ for all $t \in I$. As a consecuence, $W(f_1, f_2, \dots, f_n)(t_0) \neq 0$ for some $t_0 \in I$ implies that $f_1, f_2, \dots, f_n$ are linearly independent on I.*

Remark 4. *(a) The converse of the above theorem is not true. For example, let $f_1(t) = t|t|$ and $f_2(t) = t^2$ on $t \in I = (-1, 1)$. Note that $W(f_1, f_2)(t) = 0$ for all $t \in I$ even though f_1 and f_2 are linearly independent on I.*
(b) It is possible that the Wronskian is zero at some points and nonzero at other points on an interval I. For instance, take $f_1(x) = x$ and $f_2(x) = x^2$.

We are interested in finding linearly independent solutions of second (or higher) order homogeneous linear ODEs. If we can somehow find two solutions y_1 and y_2 such that $W(y_1, y_2)(t) \neq 0$ for some $t \in I$, then y_1 and y_2 are linearly independent and hence form a basis for the solution space (for second-order). The following theorem is important for two reasons. Firstly, it tells that the Wronskian of two solutions is either identically zero on I or is never zero on I. Secondly, it gives the Wronskian (up to a constant multiple) without knowing the solutions.

Theorem 5 (Abel's Theorem). *If $y_1(t)$ and $y_2(t)$ are two solutions of $y'' + p(t)y' + q(t)y = 0$ with $p(t)$ and $q(t)$ continuous on an open interval I, then the Wronskian of y_1 and y_2 is given by*

$$W(y_1, y_2)(t) = c \exp\left(-\int_{t_0}^{t} p(t)dt\right),$$

for some constant c.

Proof. Let us write $W(t)$ for $W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_2(t)y_1'(t)$.Then

$$\begin{aligned} W'(t) &= y_1(t)y_2''(t) - y_2(t)y_1''(t) \\ &= y_1(t)[-p(t)y_2'(t) - q(t)y_2(t)] - y_2(t)[-p(t)y_1'(t) - q(t)y_1(t)] \\ &= -p(t)W(t). \end{aligned}$$

Thus, $W(t) = c \exp\left(-\int_{t_0}^t p(t)dt\right)$ for some constant c. □

Theorem 6. *Let $y_1(t)$ and $y_2(t)$ be two solutions of $y'' + p(t)y' + q(t)y = 0$ with $p(t)$ and $q(t)$ continuous on an open interval I, and let $t_0 \in I$. Then $W(y_1, y_2)(t_0) = 0$ implies that $y_1(t)$ and $y_2(t)$ are linearly dependent on I.*

Proof. $W(y_1, y_2)(t_0) = 0$ implies that there exist constant c_1 and c_2 not both zero, such that

$$\begin{aligned} c_1 y_1(t_0) + c_2 y_2(t_0) &= 0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) &= 0. \end{aligned}$$

We claim that $c_1 y_1(t) + c_2 y_2(t) = 0$ for all $t \in I$, which would imply $y_1(t)$ and $y_2(t)$ are linearly dependent on I. Consider the function $Y(t) = c_1 y_1(t) + c_2 y_2(t)$, $t \in I$. Then $Y(t)$ is a solution to the homogeneous ODE and satisfies the initial conditions $Y(t_0) = 0, Y'(t_0) = 0$. By uniqueness, $Y(t) \equiv 0$ on I. Hence we are done. □

1.1 Method of Reduction of Order

Suppose we somehow know one nonzero solution $y_1(t)$ of $y'' + p(t)y' + q(t)y = 0$. Let us suppose that $y_2(t) = u(t)y_1(t)$ is another solution for an unknown function $u(t)$. Then $y_2'(t) = u'(t)y_1(t) + u(t)y_1'(t)$ and $y_2''(t) = u''(t)y_1(t) + 2u'(t)y_1'(t) + u(t)y_1''(t)$. Substituting in the equation and using $y_1''(t) + p(t)y_1'(t) + q(t)y_1(t) = 0$, we get $u''(t)y_1(t) + u'(t)[2y_1'(t) + p(t)y_1(t)] = 0$. This is a first-order ODE in $u'(t)$ and can be solved, if $y_1(t)$ is never zero on I, to get

$$u'(t) = \frac{1}{(y_1(t))^2} \exp\left(-\int p(t)\ dt\right),$$

which can be integrated to get $u(t)$ and hence $y_2(t)$.

1.2 Homogeneous Linear ODEs with Constant Coefficients

Consider an ODE of the form $ay'' + by' + cy = 0$ with $a \neq 0$. Let us define $L(y) = ay'' + by' + cy$. In order to find the general solution of $L(y) = 0$ we need to somehow find two linearly independent solutions. It is not difficult to guess that a solution might be of the form e^{mt}. For $y = e^{mt}$, $L(y) = (am^2 + bm + c)e^{mt}$. Therefore, $y = e^{mt}$ is a solution for some m if and only if $am^2 + bm + c = 0$. We call $\boxed{am^2 + bm + c = 0}$ the *characteristic equation* corresponding to the ODE. Now there are three cases.

Case 1: The characteristic equation has two **real and distinct roots**, say m_1 and m_2. Then $y_1 = e^{m_1 t}$ and $y_2 = e^{m_2 t}$ are two linearly independent solutions. Thus, the general solution is given by

$$\boxed{y(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t}.}$$

Case 2: The characteristic equation has **equal roots**, $m_1 = m_2 = -b/2a$. In this case $y_1(t) = e^{mt}$, where $m = -b/2a$. To find another solution $y_2(t)$ let us use the method of reduction of order. Assume $y_2(t) = u(t)e^{mt}$. Then $y_2'(t) = e^{mt}[u'(t) + mu(t)]$ and $y_2''(t) = e^{mt}[u''(t) + 2mu'(t) + m^2 u(t)]$. Substituting and dividing by e^{mt}, we get

$$au''(t) + u'(t)[2am + b] + u(t)[am^2 + bm + c] = 0.$$

Using m is a repeated root, we have $am^2 + bm + c = 0$ and $2am + b = 0$. Therefore, $u''(t) = 0$ which is equivalent to $u(t) = c_1 t + c_2$. So $(c_1 t + c_2)e^{mt}$ is a solution for any c_1, c_2. In particular, $y_2(t) = te^{mt}$ is a solution. Thus, the general solution is given by

$$\boxed{y(t) = (c_1 + c_2 t)e^{mt}.}$$

Case 3: The characteristic equation has **complex conjugate roots**, say $m_1 = \lambda + i\mu$ and $m_2 = \lambda - i\mu$, where $\mu \neq 0$. In this case $e^{m_1 t}$ and $e^{m_2 t}$ are complex-valued solutions. To get real solutions, we take

$$y_1(t) = \frac{1}{2}(e^{m_1 t} + e^{m_2 t}) = e^{\lambda t}\cos(\mu t)$$

and

$$y_2(t) = \frac{1}{2i}(e^{m_1 t} - e^{m_2 t}) = e^{\lambda t}\sin(\mu t).$$

Since $\mu \neq 0$, $y_1(t)$ and $y_2(t)$ are linearly independent. Thus, the general solution is given by

$$\boxed{y(t) = e^{\lambda t}[c_1\cos(\mu t) + c_2\sin(\mu t)].}$$

1.3 Euler-Cauchy Equations

Consider an ODE of the form $at^2y'' + bty' + cy = 0$, $t > 0$ with $a \neq 0$. By looking at the form it is natural to guess a solution of the form $y = t^m$. We see that $y = t^m$, $t > 0$ is a solution if and only if $\boxed{am(m-1) + bm + c = 0}$. This is a quadratic equation and so there are three cases.

Case 1: The characteristic equation has two **real and distinct roots**, say m_1 and m_2. Then $y_1 = t^{m_1}$ and $y_2 = t^{m_2}$ are two linearly independent solutions. Thus, the general solution is given by

$$\boxed{y(t) = c_1 t^{m_1} + c_2 t^{m_2},\ t > 0.}$$

Case 2: The characteristic equation has **equal roots**, $m_1 = m_2 = m$. In this case $y_1(t) = t^m$ is a solution. To find another solution $y_2(t)$ let us use the method of reduction of order. Assume $y_2(t) = u(t)t^m$. Verify that in this case we can take $u(t) = \ln t$, so $y_2(t) = t^m \ln t$ is another solution. Thus, the general solution is given by

$$\boxed{y(t) = (c_1 + c_2 \ln t)t^m,\ t > 0.}$$

Case 3: The characteristic equation has **complex conjugate roots**, say $m_1 = \lambda + i\mu$ and $m_2 = \lambda - i\mu$, where $\mu \neq 0$. In this case $t^{m_1} = t^\lambda e^{i\mu \ln t} = t^\lambda[\cos(\mu \ln t) + i\sin(\mu \ln t)]$ and $t^{m_2} = t^\lambda[\cos(\mu \ln t) - i\sin(\mu \ln t)]$ are complex-valued solutions. As in the constant coefficients case, the real and imaginary parts give two linearly independent real-valued solutions. Thus, the general solution is given by

$$\boxed{y(t) = t^\lambda[c_1 \cos(\mu \ln t) + c_2 \sin(\mu \ln t)],\ t > 0.}$$

Remark 5. *The Euler-Cauchy equation also has a general solution defined for $t < 0$, which is obtained by simply replacing t by $-t$ in the above formulas.*

Exercise 1. *Solve the IVP:* $4t^2y'' + 4ty' - y = 0, \quad y(-1) = 0, y'(-1) = 1.$

2 Nonhomogeneous Linear ODEs

Now we shall study nonhomogeneous linear ODEs of the form

$$y'' + p(t)y' + q(t)y = r(t), \quad t \in I, \tag{2.1}$$

where $p(t), q(t)$ and $r(t)$ are continuous real-valued functions on I.

Theorem 7. *The difference of any two solutions of the nonhomogeneous linear ODE* (2.1) *is a solution of the corresponding homogeneous linear ODE.*

Proof. Let $y(t) = y_1(t) - y_2(t)$ where y_1 and y_2 are solutions of (2.1). Then

$$\begin{aligned} y'' + p(t)y' + q(t)y &= [y_1'' + p(t)y_1' + q(t)y_1] - [y_2'' + p(t)y_2' + q(t)y_2] \\ &= r(t) - r(t) = 0. \end{aligned}$$

□

Now suppose we know a particular solution y_p of the nonhomogeneous equation. The existence (under the continuity assumption) of such a solution is guaranteed by the variation of parameters method which will be discussed later. For any solution y of the nonhomogeneous equation, the previous theorem says that $y - y_p$ is a solution of the corresponding homogeneous equation. Therefore, $y - y_p = c_1y_1 + c_2y_2$ for some constants c_1 and c_2 if y_1 and y_2 are a pair of linearly independent solutions of the homogeneous equation. This gives the following theorem.

Theorem 8. *The general solution of nonhomogeneous linear ODE is given by*

$$y(t) = c_1y_1(t) + c_2y_2(t) + y_p(t),$$

where y_1 are y_2 are a pair of linearly independent solutions of the homogeneous part, and y_p is a particular solution of the actual nonhomogeneous equation.

Summary: In order to find all solutions of a second-order nonhomogeneous linear ODE we need to find two linearly independent solutions of the corresponding homogeneous equation, and also find just one solution of the nonhomogeneous equation. We have already learnt how to find two linearly independent solutions of homogeneous linear ODEs in some cases (constant coefficients and Euler-Cauchy equations, for instance). We will learn two methods to find a particular solution of nonhomogeneous equation. One is the **variation of parameters method** which is applicable in general, provided we know two linearly independent solutions of the corresponding homogeneous equation. The difficulty in this method is that one has to evaluate some indefinite integrals which might be cumbersome. The second method is the **method of undetermined coefficients** which is easier, whenever applicable. But, one must remember that the method of undetermined coefficients can be applied only when the homogeneous part is a constant coefficient equation and $r(t)$ is restricted to some special functions (polynomials, exponentials, sine or cosine, or a sum or product of these). These methods will be discussed in the lecture classes. Do enough examples to get used to these methods.