MTL 100 (CALCULUS): TUTORIAL SHEET NO. 2

Integral Calculus

1. Using definition of Riemann integral evaluate

   (a) $\int_a^b x^2 dx$ (b) $\int_a^b e^x dx$ (c) $\int_0^{\frac{\pi}{2}} \sin^2 x dx$

   (d) $\int_a^b \frac{dx}{x}$ (e)$\int_a^b (1+x)dx$.

2. Which of the following functions are integrable over the indicated intervals:

   (a) $f(x) = |x-1|; \quad x \in [0,1]$

   (b) $f(x) = \begin{cases} 0, & a \le x < c; \\ \text{k}, & x = c; \\ 1, & c < x \le b; x \in [a,b] \end{cases}$

   (c) $f(x) = \begin{cases} 1-x, & \text{for } x \text{ rational}; \\ 1+x, & \text{for } x \text{ irrational}; x \in [0,2]. \end{cases}$

   (d) $f(x) = \begin{cases} 1, & \text{for } x \text{ rational}; \\ -1, & \text{for } x \text{ irrational}; x \in [0,2]. \end{cases}$

   (e) Let f be a function from [0,1] into R, defined by

   $f(x) = \begin{cases} 0, & \text{when } x \text{ is rational}; \\ 1, & \text{when } x \text{ is irrational}. \end{cases}$

3. Derive the Mean Value Theorem of integral calculus by applying Lagrange's Mean Value Theorem to the function: $F(x) = \int_a^x f(t)dt$ where $a \le x \le b$.

4. Verify the Mean Value Theorem as applied to the following definite integrals:

   (a) $\int_0^\pi \sin x dx$ (b)$\int_0^1 e^x dx$ (c)$\int_0^4 x^2 dx$.

5. Find $F'(x)$ if

   (a) $F(x) = \int_0^{\sin x} \frac{dt}{2+t}$

   (b) $F(x) = \int_{2x-1}^{x^2+1} \sqrt{(1+\sec^2 t)}dt$

   (c) $F(x) = \int_1^{x^2} \frac{dt}{1+\sqrt{(1-t)}}$.

6. Variables x and y are related by the equation $x = \int_0^y \frac{dt}{\sqrt{1+t^2}}$. Prove that $\frac{d^2y}{dx^2} = y$.

7. Suppose f and g are continuous on $[a,b]$ and $g(x)$ does not change sign in $[a,b]$. Prove that for some c in $[a,b]$

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$$

8. Without evaluating the integrals show that

   (a) $\frac{1}{6} < \int_0^2 \frac{dx}{10+x} \leq \frac{1}{5}$

   (b) $\int_0^2 \frac{x+1}{x^2+1} dx < \int_0^2 (x+1) dx$

   (c) $\int_{-1}^1 \frac{1+x^2}{\sqrt{x^2+2}} dx < (\frac{1}{\sqrt{2}}) \int_{-1}^1 (1+x^2) dx.$

9. Let $f(x)$ be a continuous function for all real x. Express the $\lim_{x\to\infty} \frac{1}{n}[f(\frac{1}{n}) + f(\frac{2}{n}) + \cdots + f(\frac{n}{n})]$ as a definite integral and hence evaluate the limit when

   (a) $f(x) = \frac{1}{x^2+1}$; (b) $f(x) = \sin \pi x$.

10. Prove that

    (a) $\lim_{n\to\infty} (\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}) = \log_e 2$

    (b) $\lim_{n\to\infty} \frac{1}{n}[\sin(\frac{\pi}{n}) + \sin(\frac{2\pi}{n}) + \cdots + \sin(\frac{n\pi}{n})] = \frac{2}{\pi}$

    (c) $\lim_{n\to\infty} [\frac{1}{\sqrt{n^2+1^2}} + \frac{1}{\sqrt{n^2+2^2}} + \cdots + \frac{1}{\sqrt{n^2+n^2}}] = \log(1+\sqrt{2})$

    (d) $\lim_{n\to\infty} \sum_{k=1}^n \frac{n}{(n^2+k^2x^2)} = \frac{1}{x} tan^{-1} x$, x$\neq$ 0.

11. Prove that

    (a) $\lim_{h\to 0} \frac{1}{h} \int_x^{x+h} \frac{dt}{t+\sqrt{1+t^2}} = \sqrt{1+x^2} - x$

    (b) $\lim_{h\to 0} \frac{\int_0^x (\sin t)^3 dt}{x^4} = \frac{1}{4}$.

12. Show that

    (a)$\int_0^\pi \frac{x dx}{(a^2 - \cos^2 x)} = \frac{\pi^2}{2a\sqrt{a^2-1}}$, $a > 1$

    (b) $\int_0^{\frac{\pi}{2}} \frac{\sin^m x dx}{(\sin^m x + \cos^m x)} = \frac{\pi}{4}$ for all positive real values of m. Proceed to investigate the improper integral when m is negative.

    (c) $\int_0^\pi \frac{x \tan x dx}{(\sec x + \tan x)} = \pi(\frac{\pi}{2} - 1)$.

13. Find the average value of the given function over the indicated interval

    (a) $f(x) = \frac{\cos x}{(1+\sin x)^3}$, $x \in [0, \frac{\pi}{2}]$

    (b) $f(x) = \frac{(\sqrt{x}-1)^2}{\sqrt{x}}$, $x \in [1, 4]$

    (c) $f(x) = \sqrt{x}$, $x \in [0, 4]$

-END-