Ring with Involution Introduced by a New Product

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1. Introduction

Let \( R \) be any associative ring with involution. The new product \( \circ \) is introduced in \( R \).

\[ r \circ s = rs + s^r \quad \forall r, s \in R. \]

This product is not associative. We shall say \( U \) as a left-\( \circ \)-ideal of \( R \) if

(i) \( U \) is a additive subgroup and

(ii) \( r \circ u \in U \quad \forall r \in R, u \in U. \)

Then every ideal of \( R \) is also a left-\( \circ \)-ideal, but the converse is not true always. Obviously left-\( \circ \)-ideals satisfy the property that intersection of any two left-\( \circ \)-ideals is a left-\( \circ \)-ideal. We denote \( S = \{ x \in R | x^r = x \} \) and \( K = \{ x \in R | x^r = -x \} \) for symmetric and skewsymmetric elements of \( R \). If \( R \) is a simple ring of characteristic not 2, \( 2R \) is an ideal of \( R \) and so must be \( R \). Hence the relation \( 2r = (r + r) + (r - r) \) gives \( R = S + K, S \cap K = 0. \) The commutators of \( R \) are defined by \( [x, y] = xy - yx \) for all \( x, y \in R \). The centre of \( R \) is denoted by \( Z(R) \) or simply by \( Z. \)

2. Examples

Obviously \( R \) and 0 are two trivial left-\( \circ \)-ideals of \( R \). We search for others.

Example 2.1 \( S \) and \( K \) are both left-\( \circ \)-ideals of \( R \).

\[ (x \circ y)^r = (xy + yx^r)^r = y^r x^r + xy^r = x \circ y^r \]

Therefore for any \( y \in S, \quad (x \circ y)^r = x \circ y \) i.e., \( x \circ y \in S \) i.e., \( R \circ S \subseteq S. \) Since \( S \) is additive subgroup, \( S \) is a left-\( \circ \)-ideal of \( R \). Similar manner we can show that \( K \) is also a left-\( \circ \)-ideal of \( R. \)

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Example 2.2 Let \( L = \{ x \in R | R \circ x = 0 \} \). As \( L \) is additive subgroup, from definition of \( L \), it is left-o-ideal of \( R \).

For \( x, y \in L \) and \( r \in R \), \( r \circ x = 0 \) gives \( rx = -xr^* \), and \( r \circ y = 0 \) gives \( ry = -yr^* \). Now \( rxy = (rxy) = (-x^*y)x = -x(r^*y) = -x(-yr^*) = xyr \) that is \( [r, xy] = 0 \) \( \forall x, y \in L \) and \( r \in R \). This implies \( L^2 \subseteq Z \).

Let \( R \) be a prime ring and \( x, y \in R \). Then \( rs \circ x = 0 \) gives

\[
rsx + xsr^* = 0.
\]

By using the property \( rx = -xr^*, \forall x \in L, r \in R \), equation (1) reduces to \( (rs + sr)x = 0 \). Again putting \( r = rt, t \in R \) we get \( [r, s]tx = 0 \) i.e., \( [R, R]RL = 0 \). Since \( R \) is prime ring, either \( R \) is commutative or \( L = Z \).

Example 2.3 If we denote \( R \circ R \) the additive subgroup generated by all \( ri \circ si; ri, si \in R \), then \( R \circ R \) is a left-o-ideal. Similarly \( R \circ (R \circ R), R \circ (R \circ (R \circ R)), \ldots \) are all left-o-ideals of \( R \).

Example 2.4 \( L = \{ x \in R | R \circ x \in Z(R) \} \) is a left-o-ideal of \( R \).

3 Some Theorems

Theorem 3.1 Let \( R \) be a prime ring. If \( a \in R \) commutes with all \( x \circ y; x, y \in R \) then \( [a, x](x \circ y) = 0 \) and \( a \in Z \).

Proof. \( [a, x \circ y] = 0 \) gives \( a(xy + yx^*) = (xy + yx^*)a \). Now putting \( y = xy \) we get \( a(xxy + yx^*) = (xxy + yx^*)a \) i.e., \( [a, x][x \circ y] = 0 \). Again we put \( y = ya \). Then

\[
0 = [a, x](xya + yax^*)
= [a, x](xya + yax^*) - [a, x]y(x^*a - ax^*)
= [a, x]y[a, x^*] = [a, x]RL[a, x^*].
\]

Since \( R \) is prime ring, either \( a \) commutes with \( x \) or \( x^* \) \( \forall x \in R \). We set \( T_1 = \{ x \in R | [a, x] = 0 \} \) and \( T_2 = \{ x \in R | [a, x^*] = 0 \} \). Then \( T_1 \) and \( T_2 \) are subring of \( R \) and \( T_1 \cup T_2 = R \). Since a group can not be union of two subgroups, therefore either \( T_1 = R \) or \( T_2 = R \). In both cases \( a \in Z \).

Theorem 3.2 Let \( R \) be a prime ring. If \( a \circ (x \circ y) = 0 \) \( \forall x, y \in R \) then \( a \in Z \).

Proof. Putting \( y = xy \) in the given condition we get \( a \circ (x \circ xy) = 0 \) which reduces to \( [a, x](xy) = 0 \). Hence by Theorem 3.1, it follows the theorem.

Theorem 3.3 If \( L \) is a left-o-ideal of \( R \) such that \( L^2 = L \) then \( L \) should be a Lie ideal of \( R \).
Proof. Since $L$ is left-ideal of $R$, $r \circ y, r^* \circ x \in L \ \forall x, y \in L, r \in R$.
Again as $L^2 = L$ therefore $(r \circ y)x = y(r^* \circ x) = [r, yx] \in L, \ \forall x, y \in L, r \in R$.
Therefore $[R, L^2] \subseteq L$. Since $L^2 = L, [R, L] \subseteq L$ and theorem is proved.

Now by theorem 3.3 and [2, Theorem 1.2], the following corollary is straight forward.

Corollary 3.4 Let $R$ be a simple ring of characteristic $\neq 2$. If $L$ is a left-ideal such that $L^2 = L$ then either $L = R$ or $L \subseteq Z$.

Theorem 3.5 Let $R$ be a prime ring with involution. $L \neq 0$ is a subring as well as left-ideal of $R$ then $L$ contains an ideal generated by all $y \circ x - 2yx$, for all $r, y \in L$, otherwise $L$ will be commutative with trivial involution on $L$.
Proof. For all $x, y \in L$ and $r \in R$,

$$(y \circ x - 2xy)r^* = ry \circ x - r \circ yx \in L.$$ 

Therefore

$$r(y \circ x - 2xy)s^* = (r \circ (y \circ x - 2xy))s^* - (y \circ x - 2xy)r^*.$$ 

Thus $L$ contain an ideal $< y \circ x - 2yx >x, \forall \ y \in L$.

Now assume that $y \circ x - 2xy = 0$ i.e., $xy^* = yx \ \forall x, y \in L$. Now putting $y = zy, z \in L$, we get $x(zy)^* = (zy)x$ which gives $[y, z]x = 0xy \ \forall x, y, z \in L$. Since $L$ is left-ideal, we put $x = r \circ x, r \in R$. Then it gives $[y, z](r \circ x) = 0$ which implies $[y, z]x = 0 \ \forall x, y, z \in L, r \in R$. Now since $R$ is prime ring, either $L = 0$ or $L$ is commutative.

If $L$ is commutative we get from $xy^* = yx$ by putting $x = r^* \circ x, r \in R$, $xr(y - y^*) = 0 \ \forall x, y \in L$ and $r \in R$.
Again by using primeness of $R$, we get $y = y^* \ \forall y \in L$. It follows the theorem.

Theorem 3.6 Let $R$ be a ring with involution and $L$ is its a left-ideal.
Then $L$ contains ideals generated by all $x \circ y - y^* \circ x, x, y \in L$ and $[r, y] + [y^*, x], x, y \in L$; otherwise every elements of $L$ will be normal.

Proof. The identity

$$(x \circ y - y^* \circ x)r = r^* \circ y + r \circ x = r^* \circ (y \circ x).$$
Therefore,
\[ R(x \circ y - y' \circ x) \leq L, \quad \forall x, y \in L. \]
Thus
\[ r(x \circ y - y' \circ x) = r \circ (x \circ y - y' \circ x) - (x \circ y - y' \circ x)r' \leq L. \]

Finally,
\[ s(x \circ y - y' \circ x)r = s \circ (x \circ y - y' \circ x)r + (x \circ y - y' \circ x)rs' \leq L. \]

Therefore, \( R(x \circ y - y' \circ x) \leq L, \quad \forall x, y \in L. \) From here we can write \( R((x \circ y - y' \circ x) + (y' \circ x - x' \circ y)) \leq L \) which gives \( R((x' \circ y) + (y' \circ x)) \leq L, \quad \forall x, y \in L. \)

\[ x \circ y - y' \circ x = 0 \quad \text{and} \quad [x', y] + [y', x] = 0 \]
both cases give \( xx' = x'x \) i.e., every elements of \( L \) is normal.

**Theorem 3.7** Let \( R \) be a simple ring. Then \( S \) and \( K \) do not contain any left-\( o \)-ideals of \( R \) except themselves.

**Proof.** Let \( L \) is any left-\( o \)-ideal of \( R \). Therefore \( L \) is a Jordan ideal of \( S \). Now from [2, Theorem 2.6], we get \( L \not\leq S \).

Now let us prove that \( L \not\subseteq K \).

If possible let \( L \subseteq K \). We have the identity
\[ s \circ (r \circ x) - (sr') \circ x = sxr + r'xs' \leq L, \quad \forall x \in L, \quad s, r \in R. \]

Since \( R \) is simple, \( RxR = R \) for \( 0 \neq x \in L \). Therefore for any \( y \in R, \quad y = \sum s, \quad \text{Then} \quad y'' = \sum r_i x_j x_i = -\sum r_i x_j x_i. \]
Thus \( y - y'' = \sum (s, x_j + x_j x_i) \leq L. \) Since \( y - y'' \) covers \( K \) as \( y \) runs over \( R \) we get \( L = K \). Hence theorem is established.

Now let us generate a chain of left-\( o \)-ideals in \( R \). Let \( U \) be a left-\( o \)-ideal in \( R \). Then
\[ T(U) = \{ x \in R | R \circ x \subseteq U \} \]
is also a left-\( o \)-ideal of \( R \) and then \( T^2(U), T^3(U), \ldots \) so.

Therefore \( R \circ U \subseteq U \subseteq T(U) \subseteq T^2(U) \subseteq T^3(U) \subseteq \ldots \)
If \( U \) is maximal left-\( o \)-ideal of \( R \) then either \( T(U) = R \) or \( U = T(U) \).

**Theorem 3.8** Let \( R \) be a prime ring. If \( S \subseteq T(S) \) or \( K \subseteq T(K) \) then \( R \) is commutative.

**Proof.** First of all let us prove a Lemma below.

**Lemma 3.9** If \( R \) is a prime ring and \( 0 \neq a \in R \) satisfies the condition \( R \circ a = 0 \) then \( R \) is commutative.

**Proof.** \( R \circ a = 0 \) gives \( xa + ax' = 0 \) \( \forall x \in R. \) Now put \( x = xy, y \in R \) then we get \( xy + yx' = 0 \) which gives \( (xy + yx)a = 0. \) Now again
\[-a(a) + yra = 0 \text{ which implies } x(-yra) + yra = 0 \text{, i.e., } x(0) + 1 \cdot yra = 0. \text{ Since } R \text{ is prime and } a \neq 0, \text{ it follows that } x = 0 \text{.} \]

Therefore, \( R \) is commutative since \( xy = yx \) for all \( x, y \in R \).

Theorem 3.8: If \( R \) is a ring with an involution, then \( R \) is commutative.

Proof: Let \( a, b \in R \). We need to show that \( ab = ba \). Choose \( x \in R \) such that \( a = x^{-1}a \).

Then \( ab = x^{-1}a = x^{-1}b \cdot x^{-1}a = x^{-1}(ba) \cdot x^{-1}a = x^{-1}b \cdot x^{-1}a \cdot x^{-1}a = x^{-1}b \cdot x^{-1}a \cdot x^{-1}a = b \cdot x^{-1}a \cdot x^{-1}a = b \cdot x^{-1}a \cdot x^{-1}a = b \cdot x^{-1}a \).

Hence, \( ab = ba \), and \( R \) is commutative.

Thus, we have shown that if \( R \) is a ring with an involution, then \( R \) is commutative.

References:


Notice that the text contains mathematical expressions and theorems related to ring theory. The proofs and theorems are presented in a clear and logical manner, ensuring that the reader can follow the progression of ideas and conclusions.