Units in finite loop algebras of RA2 loops

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Let $F[L]$ be the loop algebra of a loop $L$ over a field $F$. In this paper, we characterize the structure of the unit loop of $F[L]$ modulo its Jacobson radical when $L = M(D_{2m}, 2)$ is an RA2 loop obtained from the dihedral group of order $2m$, $m$ is an odd number and $F$ is a finite field of characteristic 2. The structure of $1 + J(F[L])$ is also determined.

Keywords: Loop algebra; RA2 loop, Zorn’s algebra; unit loop; general linear loop.

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1. Introduction

Let $R$ be an associative and commutative ring with unity $1 \neq 0$. A loop ring $R[L]$ of a loop $L$ can be constructed precisely in the same manner as the group ring $R[G]$ is constructed from a group $G$. If $R = F$, a field, then we call $F[L]$ a loop algebra. A ring is said to be an alternative ring if it satisfies

the left alternative identity: $x(xy) = x^2y$

and

the right alternative identity: $(yx)x = yx^2$.

A loop $L$ whose loop ring $R[L]$ over some commutative, associative ring $R$ with unity and of characteristic different from 2 is alternative, but not associative is called a Ring Alternative (RA) loop. An RA2 loop is a loop whose loop ring is alternative only when the characteristic of $R$ is 2. The problem of determining the structure of the unit group of a group ring is of much interest for many authors. But a few are trying to characterize the structure of the unit loop of a loop ring and in fact the units in finite loop algebras.
In 1992, Goodaire [3] determined the loop of units in the integral alternative loop rings of the six smallest order loops. In 1993, Jespers and Leal [5] studied the unit loop \( U(M(Q_8, 2)) \), where \( M(Q_8, 2) \) denotes the Moufang Loop obtained from \( Q_8 \), the quaternion group of order 8. The semisimple loop algebras of \( RA \) loops have been studied by Ferraz, Goodaire and Milles [2]. The structure of the unit loops of finite loop algebras of \( RA \) loops of order 32, 64 and in general of seven non-isomorphic classes of indecomposable \( RA \) loops have been determined by authors in [8–10]. But the problem of characterizing the structure of the unit loops of loop algebras of \( RA \) loops over finite fields is still open. Chein and Goodaire [1] proved that if \( G = D_{2m}, \) a dihedral group of order \( 2m \), then the loop \( M(G, 2) \) is an \( RA \) loop. In this paper, we determine the structure of the unit loop of \( F[L]/J(F[L]) \), when \( L = M(D_{2m}, 2) \), \( m \) odd number and \( F \) is a finite field of characteristic 2.

We begin by establishing some notations. \( M(G, *, g_0) \) denotes the Moufang loop obtained from the non-abelian group \( G, g_0 \in Z(G) \), the center of group \( G \), and * the involution on \( G \). Throughout the paper, \( F = F_{2^n} \) is a finite field containing \( 2^n \) elements. Also, we use the following notations:

- \( J(F[L]) \) : Jacobson radical of an alternative loop algebra \( F[L] \)
- \( C_m \) : cyclic group of order \( m \)
- \( F^* \) : \( F \setminus \{0\} \)
- \( F_{q^k} \) : field extension of \( F \) of degree \( k \)
- \( \Phi_m(x) \) : \( m \)th cyclotomic polynomial
- \( \phi(n) \) : Euler’s phi function

The following is the main result of this paper.

**Theorem 1.1.** Let \( q = 2^n, L = M(D_{2m}, 2) \) be \( RA \) loop, and \( F[L] \) be its loop algebra. Then

\[
U(F[L]/J(F[L])) \cong F^* \times \bigotimes_{d | m, d > 1} (GLL(2, F_{q^{k_d}})) \Phi_{k_d}^{(2n)},
\]

where

\[
e_d = \begin{cases} k_d/2 & \text{if } k_d \text{ is even and } q^{k_d/2} \equiv -1 \pmod{d}, \\ k_d & \text{otherwise.} \end{cases}
\]

Here \( k_d \) is the order of \( 2^n \) modulo \( d \) and \( 1 + J(F[L]) \cong C_2^{2n} \), an elementary abelian \( 2 \)-group of order \( 2^{2n} \).

**2. Some Preliminaries**

In this section, we discuss few results which will be useful in our work.

Zorn’s vector matrix algebra is an eight-dimensional non-associative algebra and is a generalization of matrix algebra over a ring. For any commutative and associative ring \( R \) (with unity), let \( R^3 \) denote the set of ordered triples over \( R \) and
consider the set of $2 \times 2$ matrices of the form $\begin{bmatrix} a & x \\ y & b \end{bmatrix}$, $a, b \in R$ and $x, y \in R^3$ with usual addition and the multiplication defined by

$$
\begin{bmatrix} a & x \\ y & b \end{bmatrix} \begin{bmatrix} c & z \\ w & d \end{bmatrix} = \begin{bmatrix} ac + x \cdot w & az + dx - y \times w \\ cy + bw + x \times z & bd + y \cdot z \end{bmatrix},
$$

where $\cdot$ and $\times$ denote the dot and cross products respectively in $R^3$. By this construction, we obtain an alternative algebra called Zorn’s vector matrix algebra denoted by $\mathfrak{Z}(R)$. There is a multiplicative determinant function in $\mathfrak{Z}(R)$, $\det: \mathfrak{Z}(R) \to R$ defined as

$$
\det \left( \begin{bmatrix} a & x \\ y & b \end{bmatrix} \right) = ab - x \cdot y.
$$

The loop of invertible elements, $\text{GLL}(2, R) = \{ A \in \mathfrak{Z}(R) | \det A \text{ is a unit in } R \}$, is a Moufang loop called the general linear loop. For more details, we refer the reader to [4, 13].

For an alternative ring $A$, an element $a \in A$ is said to be quasi-regular if there exists an element $b \in A$, called the quasi-inverse of $a$, such that $a + b = ab = ba$. An ideal is said to be quasi-regular if all its elements are quasi-regular. The Jacobson radical $J(A)$ of an alternative ring $A$ is defined as the largest quasi-regular ideal of $A$.

Vojtěchovský [11] gave the presentation of Moufang loops of the type $M(G, 2)$ with $G$ a finite two generated group, as follows.

**Theorem 2.1 ([11, Theorem 3.1]).** Let $G = \langle x, y | R \rangle$ be a presentation for a finite group $G$, where $R$ is a set of relations in generators $x, y$. Then $M(G, 2)$ is presented by

$$
\langle x, y, u | R, u^2 = (xu)^2 = (yu)^2 = (xy \cdot u)^2 = e \rangle,
$$

where $e$ is the neutral element of $G$.

To prove the main theorem, we need the following lemmas.

**Lemma 2.2.** Let $X = \begin{bmatrix} a & (0, u, 0) \\ (0, v, 0) & b \end{bmatrix}$ be an element in $\text{GLL}(2, R)$ and $A = \begin{bmatrix} c & (0, 0, 0) \\ (0, 0, 0) & d \end{bmatrix} \in \mathfrak{Z}(R)$. Then

1. $(XA)^{-1}X = X(A^{-1}X)$; that is, $XAX^{-1}$ is unambiguous.
2. If $A' = \begin{bmatrix} a & (0, 0, 0) \\ (0, v, 0) & f \end{bmatrix}$, then $X(AA')^{-1}X = (XAX^{-1})(XAX^{-1})$.

**Proof.** Let $X = \begin{bmatrix} a & (0, u, 0) \\ (0, v, 0) & b \end{bmatrix}$. Therefore, $X^{-1} = \frac{1}{ab + uv} \begin{bmatrix} b & (0, u, 0) \\ (0, v, 0) & a \end{bmatrix}$, where $0 \neq ab + uv \in F$.

1. A simple calculation yields the result.
Lemma 2.3 ([7, Lemma 3.1]). Let \( q = 2^n \), \( \gcd(q,d) = 1 \), \( \zeta \) be a primitive \( d \)th root of unity over \( F \) and let the order of \( q \) modulo \( d \) be \( k_d \). Then \( \zeta \) and \( \zeta^{-1} \) are conjugates over \( F \) if and only if \( k_d \) is even and \( q^{k_d/2} \equiv -1 \pmod{d} \). Further,

\[
[F(\zeta + \zeta^{-1}) : F] \equiv \begin{cases} 
\frac{k_d}{2} & \text{if } k_d \text{ is even and } q^{k_d/2} \equiv -1 \pmod{d}, \\
k_d & \text{otherwise.}
\end{cases}
\]

3. Proof of the Main Theorem

Proof of the Main Theorem. From [6, Theorem 2.47(ii)], it is known that the \( r \)th cyclotomic polynomial can be factorized as a product of irreducible polynomials over the field \( F \). So we can write, \( \Phi_r(x) = f_{r,1}(x)f_{r,2}(x)\ldots f_{r,t_r}(x) \), where \( f_{r,1}(x), f_{r,2}(x), \ldots, f_{r,t_r}(x) \) are irreducible polynomials over \( F \) each of degree \( k_r \) and \( t_r = \frac{\phi(r)}{r} \).

For each divisor \( d(>1) \) of \( m \), assume that

\[
A_d = \begin{cases} 
\frac{t_d}{2} & \text{if } k_d \text{ is even and } q^{k_d/2} \equiv -1 \pmod{d}, \\
t_d & \text{otherwise,}
\end{cases}
\]

and \( \lambda_{d,i} \) is a root of the irreducible factor \( f_{d,i}(x) \) over \( F \) for each \( i, 1 \leq i \leq t_d \).

Let \( D_{2m} \) be presented as \( (a,b) \mid a^m = b^2 = 1, ba = a^{-1}b \).

For \( d \mid m, d > 1 \) and \( 1 \leq i \leq t_d \), let

\[
S_{d,i} : D_{2m} \to \text{GL}(2, F(\lambda_{d,i} + \lambda_{d,i}^{-1}))
\]
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be the group homomorphism given by the assignment

\[ a \mapsto \begin{bmatrix} 0 & 1 \\ 1 & \lambda_{d,i} + \lambda_{d,i}^{-1} \end{bmatrix}, \]
\[ b \mapsto \begin{bmatrix} 1 & 0 \\ \lambda_{d,i} + \lambda_{d,i}^{-1} & 1 \end{bmatrix}. \]

From [12, Sec. 2.3], the following assignment gives the matrix representation of 
\( L = M(D_{2m}, 2) \). Now

\[ T_{d,i} : L \rightarrow \text{GL}(2, F(\lambda_{d,i} + \lambda_{d,i}^{-1})) \]
defined by

\[ a \mapsto \begin{bmatrix} 0 & (0, 1, 0) \\ (0, 1, 0) & \lambda_{d,i} + \lambda_{d,i}^{-1} \end{bmatrix}, \]
\[ b \mapsto \begin{bmatrix} 1 & (0, 0, 0) \\ (0, \lambda_{d,i} + \lambda_{d,i}^{-1}, 0) & 1 \end{bmatrix} \]

and

\[ u \mapsto \begin{bmatrix} 0 & (0, 0, 1) \\ (0, 0, 1) & 0 \end{bmatrix} \]

is a well-defined loop homomorphism.

If we take \( X_{d,i} = \begin{bmatrix} 1 \\ (0, \lambda_{d,i}, 0) \end{bmatrix} \), then \( T_{d,i}(a) = X_{d,i} \tilde{E}_{d,i} X_{d,i}^{-1} \), where \( \tilde{E}_{d,i} = \begin{bmatrix} \lambda_{d,i} & (0, 0, 0) \\ (0, \lambda_{d,i} + \lambda_{d,i}^{-1}) & 1 \end{bmatrix} \) and \( X_{d,i} \tilde{E}_{d,i} X_{d,i}^{-1} \) is unambiguous by Lemma 2.2.

By the same lemma, using induction, we get that

\[ (X_{d,i} \tilde{E}_{d,i} X_{d,i}^{-1})^r = X_{d,i}(\tilde{E}_{d,i})^r X_{d,i}^{-1} \]
for every \( r \in \mathbb{N} \).

Therefore, for all \( 1 \leq r \leq m - 1 \),

\[ T_{d,i}(a^r) = T_{d,i}(a)^r = X_{d,i}(\tilde{E}_{d,i})^r X_{d,i}^{-1} \]
\[ = \frac{1}{\lambda_{d,i} + \lambda_{d,i}^{-1}} \begin{bmatrix} \lambda_{d,i}^{-1} + \lambda_{d,i}^{-r+1} & (0, \lambda_{d,i} + \lambda_{d,i}^{-r+1}, 0) \\ (0, \lambda_{d,i} + \lambda_{d,i}^{-r+1}, 0) & \lambda_{d,i}^{-1} + \lambda_{d,i}^{-r+1} \end{bmatrix}, \]
\[ T_{d,i}(a^r b) = T_{d,i}(a^r) T_{d,i}(b) \]
\[ = \frac{1}{\lambda_{d,i} + \lambda_{d,i}^{-1}} \begin{bmatrix} \lambda_{d,i}^{-1} + \lambda_{d,i}^{-r+1} & (0, \lambda_{d,i} + \lambda_{d,i}^{-r+1}, 0) \\ (0, \lambda_{d,i} + \lambda_{d,i}^{-r+1}, 0) & \lambda_{d,i}^{-1} + \lambda_{d,i}^{-r+1} \end{bmatrix} \]
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\[ T_{d,i}(a^ru) = T_{d,i}(a^r)T_{d,i}(u) \]
\[ = \frac{1}{\lambda_{d,i} + \lambda_{d,i}^{-1}} \times \begin{bmatrix} 0 & (\lambda_{d,i}^{-r} + \lambda_{d,i}^{-r+1}) \\ (\lambda_{d,i}^{r} + \lambda_{d,i}^{-r}, 0, \lambda_{d,i}^{r+1} + \lambda_{d,i}^{-r-1}) & 0 \end{bmatrix}, \]
and
\[ T_{d,i}(a^rb - u) \]
\[ = T_{d,i}(a^rb)T_{d,i}(u) \]
\[ = \frac{1}{\lambda_{d,i} + \lambda_{d,i}^{-1}} \times \begin{bmatrix} 0 & (\lambda_{d,i}^{r+2} + \lambda_{d,i}^{-r-2}, 0, \lambda_{d,i}^{r+1} + \lambda_{d,i}^{-r-1}) \\ (\lambda_{d,i}^{r} + \lambda_{d,i}^{-r}, 0, \lambda_{d,i}^{r+1} + \lambda_{d,i}^{-r-1}) & 0 \end{bmatrix}. \]

Let
\[ T_0 : L \rightarrow F^* \]
defined by
\[ a \mapsto 1, \quad b \mapsto 1, \quad u \mapsto 1 \]
be the loop homomorphism.

If \( k_d \) is even, then define \( T = \prod_{d|m} T_d \), where \( T_d = \prod_{i=1}^{t_d} T_{d,i} \).

If \( k_d \) is odd, then \( \lambda_{d,i} \) and \( \lambda_{d,i}^{-1} \) are roots of different irreducible factors of \( \Phi_d(x) \).

Without loss of generality, we can choose that \( \lambda_{d,i}^{r+1} = \lambda_{d,i}^{-1} \) for all \( 1 \leq i \leq \frac{t_d}{2} \).

For this case, define \( T = \prod_{d|m} T_d \), where \( T_d = \prod_{i=1}^{t_d} T_{d,i} \).

Therefore
\[ T : L \rightarrow F^* \times \bigotimes_{d|d,m} GLL(F(\lambda_{d,i} + \lambda_{d,i}^{-1})) \]
is a loop homomorphism.

Let
\[ T_{d,i}^* : F[L] \rightarrow \mathfrak{g}(F(\lambda_{d,i} + \lambda_{d,i}^{-1})) \]
be the loop algebra homomorphism obtained by extending \( T_{d,i} \) linearly over \( F \). Then
\[ T^* : F[L] \rightarrow F \bigoplus \bigoplus_{d|m} \mathfrak{g}(F(\lambda_{d,i} + \lambda_{d,i}^{-1})) \]
is defined as

\[ T^* := T_0^* \oplus \bigoplus_{d, m = 1}^{A_d} T_{d, i}^*, \]

To determine the kernel of \( T^* \), consider

\[ Z = \sum_{j=0}^{m-1} \alpha_j a^j + \sum_{j=0}^{m-1} \beta_j a^j b + \sum_{j=0}^{m-1} \gamma_j a^j u + \sum_{j=0}^{m-1} \delta_j a^j bu \in \ker T^*, \]

where

\[ G_1(x) = \sum_{j=0}^{m-1} \alpha_j x^j, \quad G_2(x) = \sum_{j=0}^{m-1} \beta_j x^j, \]

\[ G_3(x) = \sum_{j=0}^{m-1} \gamma_j x^j, \quad G_4(x) = \sum_{j=0}^{m-1} \delta_j x^j \in F[x]. \]

Thus

\[ T_0^*(Z) = \sum_{j=0}^{m-1} \alpha_j + \sum_{j=0}^{m-1} \beta_j + \sum_{j=0}^{m-1} \gamma_j + \sum_{j=0}^{m-1} \delta_j. \]

For \( d \mid m, d > 1 \) and \( 1 \leq i \leq A_d \),

\[ T_{d, i}^* \left( \sum_{j=0}^{m-1} \alpha_j a^j \right) \]

\[ = \frac{1}{\lambda_{d, i} + \lambda_{d, i}^{-1}} \left[ \lambda_{d, i} G_1(\lambda_{d, i}) + \lambda_{d, i}^{-1} G_1(\lambda_{d, i}^{-1}) \right] \]

\[ \begin{bmatrix} 0, & G_1(\lambda_{d, i}) + G_1(\lambda_{d, i}^{-1}), & 0 \end{bmatrix}, \]

\[ T_{d, i}^* \left( \sum_{j=0}^{m-1} \beta_j a^j b \right) \]

\[ = \frac{1}{\lambda_{d, i} + \lambda_{d, i}^{-1}} \left[ \lambda_{d, i} G_2(\lambda_{d, i}) + \lambda_{d, i}^{-1} G_2(\lambda_{d, i}^{-1}) \right] \]

\[ \begin{bmatrix} 0, & G_2(\lambda_{d, i}) + G_2(\lambda_{d, i}^{-1}), & 0 \end{bmatrix}, \]

\[ T_{d, i}^* \left( \sum_{j=0}^{m-1} \gamma_j a^j u \right) \]

\[ = \frac{1}{\lambda_{d, i} + \lambda_{d, i}^{-1}} \left[ \lambda_{d, i} G_3(\lambda_{d, i}) + \lambda_{d, i}^{-1} G_3(\lambda_{d, i}^{-1}) \right] \]

\[ \begin{bmatrix} 0, & G_3(\lambda_{d, i}) + G_3(\lambda_{d, i}^{-1}), & 0, \end{bmatrix} \]

\[ \lambda_{d, i} G_3(\lambda_{d, i}) + \lambda_{d, i}^{-1} G_3(\lambda_{d, i}^{-1}) \]

\[ \lambda_{d, i} G_3(\lambda_{d, i}) + \lambda_{d, i}^{-1} G_3(\lambda_{d, i}^{-1}) \].
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and

\[ T_{d,i}^* \left( \sum_{j=0}^{m-1} \delta_j a_j b_j \right) \]

\[ = \frac{1}{\lambda_{d,i} + \lambda_{d,i}^{-1}} \begin{bmatrix}
0 & (\lambda_{d,i}^2 G_d(\lambda_{d,i}) + \lambda_{d,i}^{-2} G_d(\lambda_{d,i}^{-1}), 0, \\
\lambda_{d,i} G_d(\lambda_{d,i}) + \lambda_{d,i}^{-1} G_d(\lambda_{d,i}^{-1}), 0, \\
\lambda_{d,i} G_d(\lambda_{d,i}) + \lambda_{d,i}^{-1} G_d(\lambda_{d,i}^{-1})
\end{bmatrix}. \]

Therefore \( T^*(Z) = 0 \) implies that

\[ \sum_{j=0}^{m-1} \alpha_j + \sum_{j=0}^{m-1} \beta_j + \sum_{j=0}^{m-1} \gamma_j + \sum_{j=0}^{m-1} \delta_j = 0 \quad (1) \]

and for \( d \mid m, d > 1 \) and \( 1 \leq i \leq A_d \), we get

\[ \lambda_{d,i}^{-1} G_1(\lambda_{d,i}) + \lambda_{d,i} G_1(\lambda_{d,i}^{-1}) + \lambda_{d,i} G_2(\lambda_{d,i}) + \lambda_{d,i}^{-1} G_2(\lambda_{d,i}^{-1}) = 0, \]

\[ G_1(\lambda_{d,i}) + G_1(\lambda_{d,i}^{-1}) + G_2(\lambda_{d,i}) + G_2(\lambda_{d,i}^{-1}) = 0, \]

\[ G_1(\lambda_{d,i}) + G_2(\lambda_{d,i}^{-1}) + \lambda_{d,i}^2 G_2(\lambda_{d,i}) + \lambda_{d,i}^{-2} G_2(\lambda_{d,i}^{-1}) = 0, \]

\[ \lambda_{d,i} G_1(\lambda_{d,i}) + \lambda_{d,i}^{-1} G_1(\lambda_{d,i}^{-1}) + \lambda_{d,i} G_2(\lambda_{d,i}) + \lambda_{d,i}^{-1} G_2(\lambda_{d,i}^{-1}) = 0, \]

\[ G_2(\lambda_{d,i}) + G_2(\lambda_{d,i}^{-1}) + G_1(\lambda_{d,i}) + G_1(\lambda_{d,i}^{-1}) = 0, \]

\[ G_3(\lambda_{d,i}) + G_3(\lambda_{d,i}^{-1}) + \lambda_{d,i} G_3(\lambda_{d,i}) + \lambda_{d,i}^{-1} G_3(\lambda_{d,i}^{-1}) = 0, \]

\[ \lambda_{d,i} G_3(\lambda_{d,i}) + \lambda_{d,i}^{-1} G_3(\lambda_{d,i}^{-1}) + \lambda_{d,i} G_4(\lambda_{d,i}) + \lambda_{d,i}^{-1} G_4(\lambda_{d,i}^{-1}) = 0. \]

The set of Eqs. (2) imply that

\[ G_1(\lambda_{d,i}) = G_1(\lambda_{d,i}^{-1}) = G_2(\lambda_{d,i}) = G_2(\lambda_{d,i}^{-1}) = 0 \]

for \( d \mid m, d > 1 \) and \( 1 \leq i \leq A_d \), because the determinant of the matrix

\[
\begin{bmatrix}
\lambda_{d,i}^{-1} & \lambda_{d,i} & \lambda_{d,i}^{-1} & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & \lambda_{d,i}^2 & \lambda_{d,i}^{-2} \\
\lambda_{d,i} & \lambda_{d,i}^{-1} & \lambda_{d,i} & \lambda_{d,i}^{-1}
\end{bmatrix}
\]

is equal to \((\lambda_{d,i} - \lambda_{d,i}^{-1})^4\) which is non-zero as \( m \) is odd.

Similarly, the set of Eqs. (3) imply that

\[ G_3(\lambda_{d,i}) = G_3(\lambda_{d,i}^{-1}) = G_4(\lambda_{d,i}) = G_4(\lambda_{d,i}^{-1}) = 0. \]

Thus \( \lambda_{d,i} \) and \( \lambda_{d,i}^{-1} \) are roots of \( G_l(x) \) for all \( l = 1, 2, 3, 4 \), for all \( d \mid m, d > 1 \) and \( 1 \leq i \leq A_d \). So for all \( d \mid m, d > 1 \), every irreducible factor \( \Phi_d(x) \) divides \( G_l(x) \)
for all \( l = 1, 2, 3, 4 \). Since all factors of \( \Phi_d(x) \) are relatively co-prime, therefore, \( 1 + x + x^2 + \cdots + x^{m-1} \) divides \( G_l(x) \) for all \( l = 1, 2, 3, 4 \).

Hence
\[
\alpha_r = \alpha, \quad \beta_r = \beta, \quad \gamma_r = \gamma \quad \text{and} \quad \delta_r = \delta \quad \text{for all} \quad 0 \leq r \leq m - 1.
\]

Equation (1) implies
\[
\text{dim}(J) = \delta
\]

so that \( \text{dim}(F_{\lambda_d}) \leq \delta \leq 1 \).

Thus \( \ker T^* = Ff_1 + Ff_2 + Ff_3 \).

Since the characteristic of \( F \) is 2, therefore \( f_1^2 = 0, f_2^2 = 0 \) and \( f_3^2 = 0 \). Also for \( 1 \leq i, j \leq 3, f_i \) and \( f_j \) commute as
\[
f_if_j = \sum_{i=0}^{m-1} a^i + \sum_{t=0}^{m-1} a^t b + \sum_{t=0}^{m-1} a^t u + \sum_{t=0}^{m-1} a^t bu.
\]

It follows that every element of \( \ker T^* \) is nilpotent element of nilpotency index 2.

Consequently, every element is quasiregular with quasi-inverse as itself. Thus \( \ker T^* \) is a quasiregular ideal of \( F[L] \), which implies that \( \ker T^* \subseteq J(F[L]) \).

We claim that the dimension of \( F \oplus (\bigoplus_{d|m} \bigoplus_{i=1} A_d \bigoplus_{i=1} 3(F(\lambda_{d,i} + \lambda_{d,i}^{-1}))) \) over \( F \) is
\[
4m - 3.
\]

By Lemma 2.3, we find that for all \( i, 1 \leq i \leq A_d \),
\[
[F(\lambda_{d,i} + \lambda_{d,i}^{-1}) : F] = \begin{cases} 
\phi(d) / t_d & \text{if } k_d \text{ is even and } q^{k_d/2} \equiv -1 \pmod{d}, \\
8\phi(d) / t_d & \text{otherwise},
\end{cases}
\]

so that \( \dim_F(\bigoplus_{i=1} A_d \bigoplus_{i=1} 3(F(\lambda_{d,i} + \lambda_{d,i}^{-1}))) = 4\phi(d) \).

Thus
\[
\dim_F \left( F \oplus \bigoplus_{d|m} \bigoplus_{i=1} A_d \bigoplus_{i=1} 3(F(\lambda_{d,i} + \lambda_{d,i}^{-1})) \right) = 1 + 4 \sum_{d|m} \phi(d) = 1 + 4(m - 1) = 4m - 3.
\]

This implies that \( T^* \) is onto and hence \( J(F[L]) \subseteq \ker T^* \).
Thus
\[
F[L]/J(F[L]) \cong F \oplus \bigoplus_{d \mid m}^{A_d} 3(F(\lambda_d, i + \lambda_d^{-1}))
\]

\[
= \begin{cases} 
F \oplus \bigoplus_{d \mid m}^{A_d} 3(F(k_d/2))^{\frac{\phi(d)}{2}} & \text{if } k_d \text{ is even and } q^{k_d/2} \equiv -1 \pmod{d} \\
F \oplus \bigoplus_{d \mid m}^{A_d} 3(F(k_d))^{\frac{\phi(d)}{2}} & \text{otherwise} 
\end{cases}
\]

\[
\cong F \oplus \bigoplus_{d \mid m}^{A_d} 3(F(e_d))^{\frac{\phi(d)}{2}}
\]

where
\[
e_d = \begin{cases} 
k_d/2 & \text{if } k_d \text{ is even and } q^{k_d/2} \equiv -1 \pmod{d}, \\
k_d & \text{otherwise.}
\end{cases}
\]

Consider \(V_1 = 1 + J(F[L])\). An element \(h\) of \(V_1\) is of the form \(h = 1 + a_1f_1 + a_2f_2 + a_3f_3\), where \(a_i \in F\). Also \(f_i\) and \(f_j\) commute for all \(1 \leq i, j \leq 3\). This implies that \(V_1\) is a commutative loop.

Note that
\[
(f_if_j)f_k = f_i(f_jf_k) = 0 \quad \text{for all } i, j, k = 1, 2, 3.
\]

Thus \(V_1\) is an abelian group. Further observe that \(h^2 = 1\) for all \(h \in V_1\), and hence \(V_1 \cong (C_2 \times C_2 \times C_2)^n\). \(\square\)

References


