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Restricted Perfect Group Rings

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A ring $R$ is a restricted right perfect ring if every proper homomorphic image of $R$ is right perfect. A complete characterization of restricted right perfect group rings $RG$ has been obtained when the f.c. center of the group $G$ is nontrivial. The f.c. center of a group $G$ is the set of all elements of $G$ that have finitely many conjugates in $G$.

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1. INTRODUCTION

Throughout this article, $R$ is an associative ring with identity $1 \neq 0$, $J(R)$ is the Jacobson radical of $R$, and $G$ is a nontrivial group. A ring $R$ is said to be as follows: (1) semilocal if $R/J(R)$ is artinian; (2) right (left) perfect if $R$ is semilocal and $J(R)$ is right (left) T-nilpotent; (3) semiprimary if $R$ is semilocal and $J(R)$ is nilpotent. A ring $R$ is called restricted right perfect (RRP) if every proper homomorphic image of $R$ is right perfect. Similarly, we define restricted semiprimary (RSP) and restricted right artinian (RRA) rings [6]. Alberto Facchini and Catia Parolin [3] have studied noncommutative RRP rings under the name of right almost perfect rings.

Every right perfect ring is RRP, but the converse is not true. For example, $\mathbb{Z}$, the ring of integers, is a restricted perfect ring, but it is not a perfect ring.

Group ring is denoted by $RG$. If $R$ is commutative, then $RG$ is called a group algebra. If $H$ is a subgroup of $G$, then $\omega H$ will denote the right ideal of $RG$ generated by $\{1-h|h \in H\}$. In particular, if $H$ is a normal subgroup of $G$, then $\omega H$ is a two-sided ideal of $RG$ and $RG/\omega H \cong R(G/H)$. If $H = G$, then $\omega G$ is the augmentation ideal of $RG$. Then $\omega G$ is the kernel of the augmentation map, $\omega : RG \to R$, and $RG/\omega G \cong R$. If $I$ is an ideal of $R$, then $IG$ is an ideal of $RG$ and $RG/IG \cong (R/I)G$. The set $\Delta(G) = \{x \in G|x$ has finitely many conjugates in $G\} = \{x \in G|G:C_G(x) < \infty\}$ is called the f.c. center of $G$. Throughout this article, $Z(G)$ will denote the center of $G$. We refer to Passman [8] for group rings, Lam [5] for rings.

In Section 2, necessary conditions for $RG$ to be RRP, but not right perfect have been obtained. In Section 3, a complete characterization of RRP group rings...
is obtained with conditions on \( Z(G) \) and \( \Delta(G) \). In Section 4, necessary conditions for a group algebra to be RRP ring are obtained irrespective of conditions imposed on \( Z(G) \) or \( \Delta(G) \). Finally, in Section 5, examples have been given to show that if \( \Delta(G) = \{1\} \), then the necessary conditions obtained in Sections 2 and 4 are not the sufficient conditions for \( RG \) to be RRP.

2. NECESSARY CONDITIONS

We begin this section with a result of Facchini and Parolin [3].

**Lemma 2.1** ([3], Theorem 3.1). *If a ring \( R \) is RRP and not a prime ring, then \( R \) is right perfect.*

**Lemma 2.2** ([10], Lemma 1). *Let \( H \) be a normal subgroup of \( G \) such that \( |G : H| < \infty \) and \( RH \) is a prime ring. Then every nonzero ideal of \( RH \) contains a nonzero \( G \)-invariant ideal.*

**Lemma 2.3** ([10], Lemma 2). *If \( RG \) is prime and \( H \) is subgroup of \( G \) with \( |G : H| < \infty \), then \( RH \) is a prime ring.*

We also have the following well-known result.

**Proposition 2.4** ([7], Proposition 2.1). *Let \( S \) be a ring with identity and \( R \) be a subring of \( S \) with the same identity. If \( R \) is a left (right) \( R \)-direct summand of \( S \), then for any right (left) ideal \( I \) of \( R \), \( IS \cap R = I \) (resp., \( SI \cap R = I \)).*

Bass ([1], Theorem P) has proved that a ring \( R \) is right perfect if and only if \( R \) satisfies the descending chain conditions on principal left ideals. This is also given in [3] as Theorem 2.1. Thus using Proposition 2.4, the lemma given below follows immediately.

**Lemma 2.5.** Let \( R \) be a subring of \( S \) with the same identity and \( R \) is a direct summand of \( S \) as right \( R \)-module. If \( S \) is right perfect, then so is \( R \).

Another well-known result by S. M. Woods is the following one.

**Lemma 2.6** ([11], Theorem). *\( RG \) is right perfect iff \( R \) is right perfect and \( G \) is finite.*

Now we apply Lemma 2.5 to RRP group rings with \( G \) having nontrivial normal subgroup \( H \).

**Theorem 2.7.** If \( RG \) is RRP and \( H \) is any nontrivial normal subgroup of \( G \), then \( RH \) is also RRP.

**Proof.** If \( RG \) is right perfect, then the result follows from Lemma 2.5.

So we consider the case when \( RG \) is RRP, but not right perfect. By Lemma 2.1, \( RG \) is prime ring. Let \( H \) be a nontrivial normal subgroup of \( G \), then \( RG/\omega H \cong R(G/H) \) is right perfect, which implies that \( G/H \) is finite. Then \( RH \) is a prime ring
by Lemma 2.3. Let $I$ be any nonzero ideal of $RH$, then by Lemma 2.2, the ideal $I$ contains a non-zero $G$-invariant ideal $L$. If we show that $RH/L$ is right perfect, then the result follows. Now if $J = RG.L = \{\sum_{finite} s_i l_i \mid s_i \in RG, l_i \in L\}$, then $J$ is a two sided ideal of $RG$ because $H$ is a normal subgroup of $G$ and $I$ contains a nonzero $G$-invariant ideal $L$. Thus $RG/J$ is right perfect. Clearly, $(RH + J)/J$ is direct summand of $RG/J$ as a right $(RH + J)/J$-module. By Lemma 2.5, $(RH + J)/J$ is right perfect. By Proposition 2.4, $J \cap RH = L$. Thus we have that $(RH + J)/J \cong RH/(J \cap RH) = RH/L$ is right perfect.

**Corollary 2.8.** If $RG$ is RRP, but not right perfect, and $H$ is any nontrivial normal subgroup of $G$, then $RH$ is also RRP, but not right perfect.

Right perfect group rings have been completely characterized by S. M. Woods [11]. So we restrict our attention only to the case when $RG$ is RRP, but not right perfect. In the next theorem, we prove necessary conditions for $RG$ to be RRP, but not right perfect ring.

**Theorem 2.9.** If $RG$ is RRP but not right perfect, then $R$ is simple artinian and $G$ is just infinite (i.e., an infinite group in which every nontrivial normal subgroup is of finite index).

**Proof.** Clearly, the augmentation map $\omega : RG \to R$ is an epimorphism. Thus $R$ is right perfect.

If $R$ is not simple and $I$ is a nonzero proper ideal of $R$, then

$$\phi : RG \to (R/I)G$$

is an epimorphism. Thus $(R/I)G$ is right perfect, and this implies that $G$ is finite (by Lemma 2.6). So $RG$ is right perfect (again by Lemma 2.6), which is a contradiction. Hence $R$ is simple. A simple right perfect ring is simple artinian.

Similarly, $G$ cannot be finite, as otherwise $RG$ would be right perfect. Now, let $H$ be a nontrivial normal subgroup of $G$. We have $RG/\omega H \cong R(G/H)$, and hence by Lemma 2.6, the index $|G : H| < \infty$.

**3. GROUP WITH $\Delta(G) \neq \{1\}$**

First, we make a simple observation about RRP rings.

**Lemma 3.1.** A ring $R$ is RRP if and only if $M_n(R)$ is RRP.

**Proof.** It is well known that the map $I \mapsto M_n(I)$ is a bijection between the sets of ideals of $R$ and $M_n(R)$. Also, $M_n(R)/M_n(1) \cong M_n(R/I)$. So the result follows from the fact that $R$ is right perfect if and only if $M_n(R)$ is right perfect. \(\square\)

We now prove the main theorems of this section.

**Theorem 3.2.** If $Z(G) \neq \{1\}$, then $RG$ is RRP, but not right perfect if and only if $R$ is simple artinian and $G = C_\infty$ (infinite cyclic group).
Proof. If $RG$ is RRP, but not right perfect, then by Theorem 2.9 $R$ is simple artinian and also $G/Z(G)$ is finite, which implies that $G'$, the commutator subgroup of $G$, is finite ([8], Lemma 4.1.4, p. 115). We claim that $G' = [1]$ because otherwise, if $G' \neq [1]$, then $RG/\omega G' \cong R(G/G')$ is right perfect and so $G/G'$ is finite; hence $G$ is finite, which is a contradiction. Thus $G' = [1]$ and so $G$ is an abelian group in which every subgroup has finite index, which implies that $G = C_\infty$ ([9], Theorem 15.1.20, p. 446).

Conversely, let $G = \langle x \rangle$ be an infinite cyclic group, then $RG$ is not right perfect by Lemma 2.6. $R$ is simple artinian, so $R = M_n(D)$ for some division ring $D$. From Lemma 3.1, $RG$ is RRP if and only if $DG$ is RRP. So, if we prove that $DG$ is RRP, then we are done. The elements of $DG$ are uniquely written as finite sum of the form $x = \sum_{i=-\infty}^{\infty} a_i x_i$ and hence $DG$ is the ring of Laurent Polynomials $D[x, x^{-1}]$, which is a Principal Ideal Domain (PID). For any nontrivial ideal $I$ of $DG$, we prove that $DG/I$ is artinian. Let

$$I_1/I \supset I_2/I \supset I_3/I \supset \cdots$$

be a descending chain of ideals in $DG/I$. We have $I \subset I_j$ and $I_{j+1} \subset I_j$, and also $I = \langle a \rangle$, $I_j = \langle a_j \rangle$ for some $a, a_j \in DG$$.

So

$$a = b_j a_j \neq 0, \quad a_{j+1} = c_j a_j$$

for $j = 1, 2, 3, \ldots$ and $b_j, c_j \in DG$.

Thus

$$a = b_j a_j = b_{j+1} a_{j+1} = b_{j+1} c_j a_j.$$ 

$DG$ is PID, so $b_j = b_{j+1} c_j$. Which implies that

$$b_1 DG \subset b_2 DG \subset b_3 DG \subset \cdots$$

$DG$ is noetherian, so $b_k DG = b_{k+1} DG = \cdots$ for some $k$. Thus $c_k, c_{k+1}, \ldots$ are units, and so $I_k = I_{k+1} = \cdots$. Thus $DG/I$ is artinian, and hence it is right perfect. \qed

Theorem 3.3. If $Z(G) = [1]$ and $\Delta(G) \neq [1]$, then $RG$ is RRP, but not right perfect if and only if $R$ is simple artinian and $G = D_\infty$ (infinite dihedral group).

Proof. Conditions on $R$ follow directly from Theorem 2.9. By Lemma 2.1, $RG$ is prime and so $\Delta(G)$ is torsion-free abelian ([2], Theorem 8). By Theorem 2.7, the group ring $R\Delta(G)$ is RRP, and so by Theorem 2.9, the f.c. center $\Delta(G)$ is just infinite. Thus $\Delta(G)$ is infinite cyclic (say $\Delta(G) = \langle x \rangle$) ([9], Theorem 15.1.20, p. 446). So, it has only one nontrivial automorphism sending $x$ to $x^{-1}$. Thus, for any $y \notin C_G(\Delta(G)) = \Delta(G)$, $y^{-1} xy = x^{-1}$, i.e., for any elements $y_1, y_2 \notin \Delta(G)$, we have $y_1^{-1} y_2 = x^{-1} = y_2^{-1} y_1$. Thus $y_1 y_2^{-1} \in C_G(x) = \Delta(G)$. i.e., $\Delta(G)y_1 = \Delta(G)y_2$. Hence $|G: \Delta(G)| = 2$ and $G = \langle x \rangle$ or $\langle x \rangle \cup \langle x \rangle y$. Now it can be easily seen that $y^2 \in Z(G) \cap \langle x \rangle$. As $Z(G) = [1]$, we get $y^2 = 1$ and $G = \langle x, y | y^2 = 1, y^{-1} xy = x^{-1} \rangle = D_\infty$. 


Conversely, without loss of generality, we can assume $R$ to be a division ring.

Let $G = \langle x, y \mid y^2 = 1, y^{-1}xy = x^{-1}\rangle$ be infinite dihedral group, and let $H = \langle x \rangle$. So we have $\Delta(G) = H$ with $|G : H| = 2$ and $RG = RH \oplus RH_y$. Let $I$ be a nonzero ideal of $RG$. Then $L = I \cap RH$ is a nonzero ideal of $RH$ ([4], Theorem 2). Now as $H$ is infinite cyclic, so by the previous theorem $RH/L$ is finite dimensional over $R$. Also, $(r_1 + L, r_2y + L) \mapsto r_1 + r_2y + I$ is an $R$-homomorphism of $RH/L \oplus (RH_y + L)/L$ onto $RG/I$. Thus $RG/I$ is finite dimensional over $R$. Therefore, $RG/I$ is artinian, and hence it is right perfect. □

4. RRP GROUP ALGEBRA

We now consider group algebra $RG$ by taking $R$ to be a commutative (simple) ring (i.e., $R$ is a field), and take into consideration the case when $\Delta(G)$ can be trivial also.

Theorem 4.1. Let $R$ be a field. If $RG$ is RRP, but not right perfect, and $G$ is any group, then one of the following equations holds:

1. $G = C_{\infty}$;
2. $G = D_{\infty}$;

Before the proof of the above theorem, we prove a result about center of $RG$. Given an element $z = \sum_{g \in G} x_g g$, we define the support of $z$ as $suppz = \{g \in G : x_g \neq 0\}$.

Lemma 4.2. Let $R$ be a commutative ring. $\Delta(G) = \{1\}$ iff $Z(RG) = R$.

Proof. Let $z = \sum_{g \in G} x_g g \in Z(RG)$. Then for all $h \in G$ we have

$$hx = xh \iff z = h^{-1}xh$$

$$\iff x_g = x_{h^{-1}gh} \quad \text{for all } h \in G \quad \text{and} \quad g \in suppz$$

Now as $\Delta(G) = \{1\}$, i.e., each conjugacy class of $g \in G$ is infinite except identity element. Thus in $z = \sum_{g \in G} x_g g$ we have all $x_g = 0$ except for $g = 1$; otherwise, the support of $z$ would be infinite. Hence, $z = \sum_{g \in G} x_g g \in Z(RG) \iff z \in R$, i.e., $Z(RG) = R$. □

Proof of Theorem 4.1. By Lemma 2.1, we have that $RG$ is prime ring. By [2] Theorem 8, either $\Delta(G)$ is torsion free abelian or $\Delta(G) = \{1\}$. If $\Delta(G)$ is torsion free abelian, then by Theorem 3.2 and Theorem 3.3, the first two assertions follow. Also if $\Delta(G) = \{1\}$, then by Theorem 4.2, the third assertion follows. □

Remark 4.3. The converse of theorem 4.1 is true in case (1) and (2) holds. But if (3) holds, then the converse is not true, counterexamples for which are given in the next section.
5. EXAMPLES

If $\Delta(G) \neq \{1\}$, then RRP group rings have been completely characterized. But if we take $\Delta(G) = \{1\}$, then the converse of Theorem 2.9 and Theorem 4.1(3) is not true, i.e., $RG$ may or may not be RRP for $G$ being a just infinite group and $R$ a simple artinian ring. First we see the case when answer is in affirmative.

Example 5.1. Let $K$ be a field and $G$ be an algebraically closed group or universal locally finite group. Then in both cases $G$ is a simple group and $\Delta(G) = \{1\}$. By [8] Corollary 9.4.6 and Corollary 9.4.10, $\omega G$ is the unique proper ideal of $KG$. As $KG/\omega G \cong K$. Thus $K$ is the only proper homomorphic image of $KG$. Thus $KG$ is RRP, but not right perfect as $G$ is infinite.

Next we see the case when answer is in negative for a group $G$ with $\Delta(G) = \{1\}$.

Example 5.2. Let $G = C_2 : C_\infty$ be the wreath product of $C_2$ by $C_\infty$, and $K$ be a field. By [8] Lemma 9.2.8, $\Delta(G) = \{1\}$ and the base group $H = \prod_i (C_2)_i$, that is the weak direct product of copies of $C_2$, is a normal subgroup of $G$ with $G/H \cong C_\infty$. So $KG/\omega H \cong KC_\infty$, which is not right perfect. Thus $KG$ is not RRP.

In Example 5.2, $\Delta(G)$ is trivial, but group $G$ is not just infinite. In the following examples, not only is $\Delta(G)$ trivial, but also $G$ is just infinite, while still $KG$ is not RRP:

Example 5.3. Let $K$ be a field of characteristic 0 and $G$ be an infinite alternating group, i.e., $G = \text{Alt}_\Omega$, where $\Omega$ is an infinite set and each element of $G$ moves only finitely many points. Clearly, $G$ is a simple group and $\Delta(G) = \{1\}$. We form the permutation module $V = \{\sum_{i \in \Omega} a_i | a_i \in K, i \in \Omega, a_i = 0$ except for finitely many $i\}$ for $KG$. $V$ has as a $K$-basis the elements of $\Omega$ and $G$ acts on $V$ by appropriately permuting this basis. If $\sigma$ and $\tau$ are two disjoint permutations in $G$, for example, take $\sigma = (i_1, i_2, i_3)$ and $\tau = (i_4, i_5, i_6)$, where $i_1, i_2, i_3, i_4, i_5, i_6$ are distinct elements. Then it can be easily seen that $(\sigma - 1)(\tau - 1) \neq 0$ and $(\sigma - 1)(\tau - 1)$ belong to the ideal $I = \text{Ann}_{KG} V$, but $(\sigma - 1) \notin I$. So, $I$ is a nontrivial proper ideal of $KG$. Clearly, the permutations $(i_1, i_2, i_3)$, $j = 3, 4, 5, \ldots$ are linearly independent mod $I$. So, $KG/I$ is infinite dimensional over $K$. Thus, $KG/I$ is not artinian. Now, as $G$ is locally finite, so $KG$ is a Von Neumann Regular ring ([2], Theorem 3). As every homomorphic image of a Von Neumann regular ring is Von Neumann regular, $KG/I$ is Von Neumann regular, and in particular $J(KG/I) = 0$. We get $(KG/I)/J(KG/I) \cong KG/I$ is not artinian. Thus $KG/I$ is not semilocal, and hence it is not right perfect. Thus $KG$ is not RRP.

Example 5.4. Let $K$ be a field of characteristic 0. Also, $G = \text{Sym}_\Omega$, where $\text{Sym}_\Omega$ denotes the group of all restricted permutations on an infinite set $\Omega$ moving only finitely many points of $\Omega$. Clearly, $G$ is locally finite and $H = \text{Alt}_\Omega$ is the only nontrivial normal subgroup of $G$ with $|G:H| = 2$. It can be observed that $KG$ is not RRP because if it had been so, then by Theorem 2.7, $KH$ would also had been RRP, which is not possible in view of Example 5.3.
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REFERENCES