THE UNIT GROUP OF $\mathbb{F}_q[D_{30}]$

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Abstract. Let $\mathbb{F}_q$ be a finite field with $q = p^n$ elements and $D_{30}$ be the dihedral group of order 30. A complete characterization of the unit group of the group algebra $\mathbb{F}_q[D_{30}]$ has been obtained.

1. Introduction. Let $F[G]$ denote the group algebra of a finite group $G$ over a field $F$. For a normal subgroup $H$ of $G$, the natural homomorphism $\omega : G \to G/H$ can be extended to an $F$-algebra epimorphism $\omega^* : F[G] \to F[G/H]$,

$$\omega^* \left( \sum_{g \in G} a_g g \right) := \sum_{g \in G} a_g \omega(g).$$

The kernel of this homomorphism is denoted by $\Delta(G, H)$. If $F[G]$ is semi-simple, then $F[G] \cong F[G/G'] \oplus \Delta(G, G')$ and by [9, Proposition 3.6.11], $F[G/G']$ is the sum of all commutative simple components of $F[G]$ and $\Delta(G, G')$ is the sum of all the non-commutative simple components.

We now discuss the famous Witt-Berman theorem. Let $F$ be a finite field of characteristic $p$. An element $g \in G$ is said to be $p$-regular if $(p, o(g)) = 1$. Let

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\end{itemize}
m be the L.C.M. of the orders of the $p$-regular elements of $G$ and $\xi$ be a primitive $m$-th root of unity over $F$. Let $T$ be the multiplicative group consisting of those integers $t$, taken modulo $m$, for which $\xi \mapsto \xi^t$ defines an automorphism of $F(\xi)$ over $F$. Two $p$-regular elements $g, h \in G$ are said to be $F$-conjugate if $g^t = x^{-1}hx$ for some $x \in G$ and $t \in T$. This defines an equivalence relation which partitions the $p$-regular elements of $G$ into $p$-regular $F$-conjugacy classes. Witt-Berman theorem [4, Ch. 17, Theorem 5.3] asserts that the number of non-isomorphic simple $F[G]$-modules is equal to the number of $F$-conjugacy classes of $p$-regular elements of $G$.

Many authors [6, 8, 10, 11] have studied the structure of the group of units of the integral group ring $\mathbb{Z}[G]$ and group algebra $F[G]$. In this paper, we restrict to the case, when $F = \mathbb{F}_q$ is a finite field with $q = p^k$ elements and give a characterization of the unit group $U(\mathbb{F}_q[D_{30}])$ of the group algebra of $D_{30}$ over $\mathbb{F}_q$. In [7], authors have generalized the results for $q = 2^k$ and given a complete characterization of units in $\mathbb{F}_{2^k}[D_{2n}]$, the group algebra of the dihedral group $D_{2n}$ of order $2n$ over the finite field $\mathbb{F}_q$, when $n$ is odd.

The dihedral group $D_{30}$ has the presentation $\langle a, b \mid a^{15} = b^2 = b^{-1}ab = a^{-1} \rangle$. The conjugacy classes of $D_{30}$ are $C_0 = \{1\}$, $C_i = \{a^i, a^{-i}\} (1 \leq i \leq 7)$ and $C_8 = \{b, ab, \cdots, a^{14}b\}$. Hence $\{C_i \mid 0 \leq i \leq 8\}$ forms an $\mathbb{F}_q$-basis of the center $\mathbb{Z}(\mathbb{F}_q[D_{30}])$ of the group algebra $\mathbb{F}_q[D_{30}]$, where $\hat{C}_i$ denotes the sum of all the elements in the conjugacy class $C_i$.

Throughout the paper, $C_n$ is the cyclic group of order $n$, $F^* = F \setminus \{0\}$, $J(F[G])$ is the Jacobson radical of the group algebra $F[G]$, $Z(F[G])$ is its center and for any $k \in \mathbb{N}$, $G^k$ denotes the external direct product of $G$ taken $k$ times. For $n \geq 2$, $M(n, F)^k$ denotes the external direct sum of $M(n, F)$ taken $k$ times.

For any group $G$ with elements $g_1, \cdots, g_{n+1}$, $(g_1, g_2) = g_1^{-1}g_2^{-1}g_1g_2$ and $(g_1, \cdots, g_{n+1}) = ((g_1, \cdots, g_n), g_{n+1})$ for $n \geq 2$. The lower central chain of $G$ is defined inductively by $\gamma_1(G) = G$ and $\gamma_{n+1}(G) = \langle \gamma_n(G), G \rangle = \langle \{x, g \mid x \in \gamma_n(G), g \in G\} \rangle$. It is easy to see that $\gamma_n(G) = \langle \{g_1, \cdots, g_n\} \mid g_i \in G\rangle$. $G$ is said to be nilpotent of class $c$ if $\gamma_{c+1}(G) = 1$ but $\gamma_c(G) \neq 1$. We shall denote by $G'$, the commutator subgroup $\gamma_2(G)$ of $G$. The commutator subgroup of $D_{30}$ is $D_{30}' = A = \langle a \rangle$.

2. Main result. In this paper, we give a complete characterization of $U(\mathbb{F}_q[D_{30}])$, the unit group of the group algebra $\mathbb{F}_q[D_{30}]$ of the dihedral group $D_{30}$ over an arbitrary finite field $\mathbb{F}_q$ containing $q$ elements.

The main result in this paper, is the following theorem.
The unit group of $\mathbb{F}_q[D_{30}]$

\textbf{Theorem 2.1.} Let $\mathbb{F}_q$ be a finite field with $|\mathbb{F}_q| = q = p^n$ and $V_1 = 1 + J(\mathbb{F}_q[D_{30}])$.

1. If $p = 2$, then $U(\mathbb{F}_q[D_{30}])$ is isomorphic to

\begin{align*}
(i) & \quad C_n^2 \times C_{q-1} \times GL(2, \mathbb{F}_q^7) & \text{if } n \equiv 0 \mod 4 \\
(ii) & \quad C_n^2 \times C_{q-1} \times GL(2, \mathbb{F}_q) \times GL(2, \mathbb{F}_q^2) \times GL(2, \mathbb{F}_q) & \text{if } n \equiv \pm 1 \mod 4 \\
(iii) & \quad C_n^2 \times C_{q-1} \times GL(2, \mathbb{F}_q)^3 \times GL(2, \mathbb{F}_q^2)^2 & \text{if } n \equiv 2 \mod 4
\end{align*}

2. If $p = 3$, then $U(\mathbb{F}_q[D_{30}])$ is isomorphic to

\begin{align*}
(i) & \quad V_1 \times (C_{q-1} \times GL(2, \mathbb{F}_q)) & \text{if } n \text{ is odd} \\
(ii) & \quad V_1 \times (C_{q-1} \times GL(2, \mathbb{F}_q^2) \times GL(2, \mathbb{F}_q)) & \text{if } n \text{ is even}
\end{align*}

and $V_1$ is a nilpotent class 2 group of exponent 3.

3. If $p = 5$, then $U(\mathbb{F}_q[D_{30}]) \cong V_1 \times (C_{q-1} \times GL(2, \mathbb{F}_q))$ and $V_1$ is a nilpotent group of class 4 and exponent 5.

4. If $p > 5$, then $U(\mathbb{F}_q[D_{30}])$ is isomorphic to

\begin{align*}
(i) & \quad C_{q-1}^2 \times GL(2, \mathbb{F}_q^7) & \text{if } q \equiv \pm 1 \mod 15 \\
(ii) & \quad C_{q-1}^2 \times GL(2, \mathbb{F}_q^2) \times GL(2, \mathbb{F}_q) & \text{if } q \equiv \pm 1 \mod 15 \\
(iii) & \quad C_{q-1}^2 \times GL(2, \mathbb{F}_q) \times GL(2, \mathbb{F}_q^2) \times GL(2, \mathbb{F}_q) & \text{if } q \equiv \pm 1 \mod 15
\end{align*}

\textbf{3. Wedderburn decomposition of $\mathbb{F}_q[D_{30}]/J(\mathbb{F}_q[D_{30}])$.} In this section, we obtain Wedderburn Decomposition of the semi-simple group algebra $\mathbb{F}_q[D_{30}]$ when $q = p^n$ and $p > 5$. We also obtain Wedderburn decomposition of $\mathbb{F}_q[D_{30}]/J(\mathbb{F}_q[D_{30}])$ when $q = p^n$ for $p = 2, 3$ and 5. Our results depend on Witt-Berman theorem.

We shall need the following lemmas in this section:

\textbf{Lemma 3.1.} Let $F$ be a finite extension of $\mathbb{F}_p$ of even degree. Then,

$$F \otimes_{\mathbb{F}_p} F_{p^2} \cong F \oplus F$$

\textbf{Proof.} By [1, Proposition 10.15, pp. 132], $F \otimes_{\mathbb{F}_p} F_{p^2} \cong \tilde{F}$ or $F \oplus F$, where $\tilde{F}$ is a quadratic extension of $F$. But $|F : \mathbb{F}_p|$ is even. Therefore, $X \in F \otimes_{\mathbb{F}_p} F_{p^2}$ gives $X^{[\tilde{F}]} = X$ showing that $F \otimes_{\mathbb{F}_p} F_{p^2} \cong \tilde{F}$ is not possible. \(\square\)
Lemma 3.2. If \( q = p^n \equiv \pm 1 \mod 5 \), then \( \mathbb{F}_q \) contains \( \sqrt{5} \).

Proof. By quadratic reciprocity, it can be seen that if \( p \equiv \pm 1 \mod 5 \),
then \( \left( \frac{5}{p} \right) = 1 \) and hence, \( \mathbb{F}_q \) contains \( \sqrt{5} \).

If \( p \equiv \pm 3 \mod 5 \), then \( \left( \frac{5}{p} \right) = -1 \), i.e., \( 5^{p-1} \equiv -1 \mod p \). Also,
\( p^n \equiv \pm 1 \mod 5 \Rightarrow n \) is even. Now, \( 5^{(p-1)} = 5^{(2^i)}(1+p+\cdots+p^{n-1}) \Rightarrow 5^{(p-1)} \equiv (-1)(1+p+\cdots+p^{n-1}) \mod p \). Since \( n \) is even, therefore \( (1 + p + \cdots + p^{n-1}) \) is even.
So, \( 5^{(p-1)} \equiv 1 \mod p \). That is, \( 5^{(p-1)} = 1 \) in \( \mathbb{F}_q \). Let \( \alpha \) be a generator of \( \mathbb{F}_q^* \). Then, as \( (p, 5) = 1 \), \( 5 = \alpha^k \) for some \( k \in \mathbb{Z} \). So, \( \alpha^{k(\frac{p-1}{2})} = 1 \) in \( \mathbb{F}_q \). Since \( o(\alpha) = p^n - 1 \), we have, \( (p^n - 1)|k \left( \frac{p^n - 1}{2} \right) \). That means, \( 2|k \) and hence \( k = 2m \) for some \( m \in \mathbb{Z} \), showing thereby that \( 5 = (\alpha^m)^2 \). Thus, \( \mathbb{F}_q \) contains \( \sqrt{5} \).

3.1. Char \( \mathbb{F}_q = p \geq 5 \). Since \( p > 5 \), therefore by Maschke’s theorem, \( \mathbb{F}_q[D_{30}] \) is semi-simple. Thus, \( J(\mathbb{F}_q[D_{30}]) = (0) \) and \( \mathbb{F}_q[D_{30}]/J(\mathbb{F}_q[D_{30}]) \cong \mathbb{F}_q[D_{30}] \). The following theorem gives the Wedderburn decomposition of \( \mathbb{F}_q[D_{30}] \) and hence that of \( \frac{\mathbb{F}_q[D_{30}]}{J(\mathbb{F}_q[D_{30}])} \) for the case when \( p > 5 \).

Theorem 3.3. If char \( \mathbb{F}_q = p > 5 \), then \( \mathbb{F}_q[D_{30}] \) is isomorphic to

(i) \( \mathbb{F}_q \oplus \mathbb{F}_q \oplus M(2, \mathbb{F}_q)^7 \) if \( q \equiv \pm 1 \mod 15 \)
(ii) \( \mathbb{F}_q \oplus \mathbb{F}_q \oplus M(2, \mathbb{F}_q)^3 \oplus M(2, \mathbb{F}_q^2)^2 \) if \( q \equiv \pm 4 \mod 15 \)
(iii) \( \mathbb{F}_q \oplus \mathbb{F}_q \oplus M(2, \mathbb{F}_q) \oplus M(2, \mathbb{F}_q^2) \oplus M(2, \mathbb{F}_q^3) \) if \( q \equiv \pm 2 \mod 15 \)

or \( q \equiv \pm 7 \mod 15 \)

Proof. Since \( p \neq 2 \), therefore, \( \mathbb{F}_q[D_{30}/D_{30}'] \cong \mathbb{F}_q[C_2] \cong \mathbb{F}_q \oplus \mathbb{F}_q \) and hence using the Wedderburn structure theorem and [9, Proposition 3.6.11], we have the following

\[
\mathbb{F}_q[D_{30}] \cong \mathbb{F}_q \oplus \mathbb{F}_q \oplus \left( \oplus_{i=1}^k M(n_i, K_i) \right)
\]

where \( n_i \geq 2 \) and \( K_i \)'s are finite dimensional division algebras over \( \mathbb{F}_q \) and hence, finite field extensions of \( \mathbb{F}_q \).

By (3.1), \( \mathcal{Z}(\mathbb{F}_q[D_{30}]) \cong \mathbb{F}_q \oplus \mathbb{F}_q \oplus \left( \oplus_{i=1}^k K_i \right) \).

For any \( r \in \mathbb{N} \), we observe that

\[
x^{q^r} = x \quad \forall \ x \in \mathcal{Z}(\mathbb{F}_q[D_{30}])
\]
The unit group of $\mathbb{F}_q[D_{30}]$  

$\iff \hat{C}_i^r = \hat{C}_i \forall 0 \leq i \leq 8$

$\iff \hat{C}_1^r = \hat{C}_1$

$\iff (a + a^{-1})^q = a + a^{-1}$

$\iff a^q + a^{-q} = a + a^{-1}$

$\iff a^q = a$ or $a^{-1}$

(3.2)

$\iff 15|q^r - 1$ or $15|q^r + 1$

Now for each $i$, $1 \leq i \leq k$, let $K_i^* = \langle y_i \rangle$.

Then

$x^q = x \forall x \in \mathcal{Z}(\mathbb{F}_q[D_{30}])$

$\iff y_i^q = y_i$

$\iff y_i^{q^r-1} = 1$

$\iff q^{[K_i:F_q]} - 1 | q^r - 1$

$\iff p^{n[K_i:F_q]} - 1 | p^{nr} - 1$

$\iff n[K_i:F_q]|nr$

(3.3)

$\iff [K_i:F_q]|r \forall i = 1, \ldots , k$

From (3.2) and (3.3), we find that the least number $t$ such that $15|q^t + 1$

or $15|q^t - 1$ is

$t = \text{l.c.m.}\{[K_i:F_q] | 1 \leq i \leq k \}$

and applying these observations to the following cases, we get,

1. $q \equiv \pm 1 \mod 15$
   $\Rightarrow t = 1$
   $\Rightarrow \text{l.c.m.}\{[K_i:F_q] | 1 \leq i \leq k \} = 1$

2. $q \equiv \pm 2$ or $\pm 7 \mod 15$
   $\Rightarrow t = 4$
   $\Rightarrow \text{l.c.m.}\{[K_i:F_q] | 1 \leq i \leq k \} = 4$

3. $q \equiv \pm 4 \mod 15$
   $\Rightarrow t = 2$
   $\Rightarrow \text{l.c.m.}\{[K_i:F_q] | 1 \leq i \leq k \} = 2$

We use the Witt-Berman theorem to find the number of simple components in the
Wedderburn decomposition of $\mathbb{F}_q[D_{30}]$. Observe that $m = 30$. Let $c$ be
the number of simple components in the Wedderburn decomposition of $\mathbb{F}_q[D_{30}]$.

Then, as discussed in the Introduction, $T$ and the $p$ regular $\mathbb{F}_q$-conjugacy classes are as follows:

1. $q \equiv 1 \mod 15$
   $T = \{1\} \mod 30$ and hence, $C_i$, $0 \leq i \leq 8$, are the $p$-regular $\mathbb{F}_q$-conjugacy classes. Hence, $c = 9$.

2. $q \equiv -1 \mod 15$
   $T = \{-1, 1\} \mod 30$ and $C_i$, $0 \leq i \leq 8$, are the $p$-regular $\mathbb{F}_q$-conjugacy classes and $c = 9$.

3. $q \equiv 2$ or $-7 \mod 15$
   $T = \{1, 17, 19, 23\} \mod 30$. We observe that, $a^{17} = a^2$, $a^{19} = a^4$, $a^{23} = a^8 = a^{-7}$, $(a^3)^{17} = a^6$ and thus, $\{1\}, \{a, a^{-1}, a^2, a^{-2}, a^4, a^{-4}, a^7, a^{-7}\}$, $\{a^3, a^{-3}, a^6, a^{-6}\}$, $\{a^5, a^{-5}\}$ and $\{b, ab, a^2b, \ldots, a^{14}b\}$ are the $p$-regular $\mathbb{F}_q$-conjugacy classes as in the previous case and $c = 5$.

4. $q \equiv -2$ or $7 \mod 15$
   $T = \{1, 7, 13, 19\} \mod 30$. Since $a^{13} = a^{-2}, a^{19} = a^4$, $(a^3)^7 = a^{21} = a^6$, therefore, $\{1\}, \{a, a^{-1}, a^2, a^{-2}, a^4, a^{-4}, a^7, a^{-7}\}$, $\{a^3, a^{-3}, a^6, a^{-6}\}$, $\{a^5, a^{-5}\}$ and $\{b, ab, a^2b, \ldots, a^{14}b\}$ are the $p$-regular $\mathbb{F}_q$-conjugacy classes as in the previous case and $c = 5$.

5. $q \equiv 4 \mod 15$
   $T = \{1, 19\} \mod 30$. Now, $a^{19} = a^4$ and $(a^2)^{19} = a^8$, the $p$-regular $\mathbb{F}_q$-conjugacy classes are $\{1\}, \{a, a^{-1}, a^4, a^{-4}\}$, $\{a^2, a^{-2}, a^7, a^{-7}\}$, $\{a^3, a^{-3}\}$, $\{a^5, a^{-5}\}$, $\{a^6, a^{-6}\}$ and $\{b, ab, a^2b, \ldots, a^{14}b\}$ and $c = 7$.

6. $q \equiv -4 \mod 15$
   $T = \{1, 11\} \mod 30$. As, $a^{11} = a^{-4}$ and $(a^2)^{11} = a^7$, the $p$-regular $\mathbb{F}_q$-conjugacy classes are $\{1\}, \{a, a^{-1}, a^4, a^{-4}\}$, $\{a^2, a^{-2}, a^7, a^{-7}\}$, $\{a^3, a^{-3}\}$, $\{a^5, a^{-5}\}$, $\{a^6, a^{-6}\}$ and $\{b, ab, a^2b, \ldots, a^{14}b\}$ as in the previous case and $c = 7$.

Since $\dim_{\mathbb{F}_q} \mathbb{Z}(\mathbb{F}_q[D_{30}])$ = number of conjugacy classes of $D_{30} = 9$, we find $\sum_{i=1}^{k} [K_i : \mathbb{F}_q] = 7$ by (3.1). Using this along with the information about l.c.m.$\{[K_i : \mathbb{F}_q] | 1 \leq i \leq k\}$ and the number of components $c(= k + 2)$ in the decomposition, we have the following possibilities for $S = ([K_i : \mathbb{F}_q])_{i=1}^{k}$ depending on $q$:
(1) If \( q \equiv \pm 1 \mod 15 \), then \( S = (1, 1, 1, 1, 1, 1) \).

(2) If \( q \equiv \pm 4 \mod 15 \), then \( S = (1, 1, 1, 2, 2) \).

(3) If \( q \equiv \pm 2 \) or \( \pm 7 \mod 15 \), then \( S = (1, 2, 4) \).

Due to the dimension constraints, \( n_i > 2 \) is not possible for any \( 1 \leq i \leq k \) and hence, \( n_i = 2 \forall 1 \leq i \leq k \) which proves that \( \mathbb{F}_q[D_{30}] \cong \)

\[
\begin{align*}
\mathbb{F}_q &\oplus \mathbb{F}_q \oplus M(2, \mathbb{F}_q)^7 & \text{if } q \equiv \pm 1 \mod 15 \\
\mathbb{F}_q &\oplus \mathbb{F}_q \oplus M(2, \mathbb{F}_q)^3 \oplus M(2, \mathbb{F}_q^2)^2 & \text{if } q \equiv \pm 4 \mod 15 \\
\mathbb{F}_q &\oplus \mathbb{F}_q \oplus M(2, \mathbb{F}_q) \oplus M(2, \mathbb{F}_q^2) & \text{if } q \equiv \pm 2 \mod 15 \\
&\oplus M(2, \mathbb{F}_q^4) & \text{if } q \equiv \pm 7 \mod 15 \square
\end{align*}
\]

3.2. Char \( \mathbb{F}_q = 2 \).

**Theorem 3.4.** If \( q = 2^n \), then,

\[
\frac{\mathbb{F}_q[D_{30}]}{J(\mathbb{F}_q[D_{30}])} \cong \begin{cases} 
\mathbb{F}_q \oplus M(2, \mathbb{F}_q)^7 & \text{if } n \equiv 0 \mod 4 \\
\mathbb{F}_q \oplus M(2, \mathbb{F}_q) \oplus M(2, \mathbb{F}_q^2) & \oplus M(2, \mathbb{F}_q^4) & \text{if } n \equiv \pm 1 \mod 4 \\
\mathbb{F}_q \oplus M(2, \mathbb{F}_q^3) \oplus M(2, \mathbb{F}_q^2) & & \text{if } n \equiv 2 \mod 4
\end{cases}
\]

**Proof.** Now, the 2 regular elements in \( D_{30} \) are \( 1, a, a^{-1}, a^2, a^{-2}, a^3, a^{-3}, a^4, a^{-4}, a^5, a^{-5}, a^6, a^{-6}, a^7, a^{-7} \). By simple calculations and Witt-Berman theorem, \( T \) and the number of simple components \( c \) in the Wedderburn decomposition of \( \mathbb{F}_q[D_{30}] \) are as follows:

(1) \( n \equiv 0 \mod 4 \)

\( T = \{1\} \mod 15 \) and \( \{1\}, \{a, a^{-1}\}, \{a^2, a^{-2}\}, \{a^3, a^{-3}\}, \{a^4, a^{-4}\}, \{a^5, a^{-5}\}, \{a^6, a^{-6}\}, \{a^7, a^{-7}\} \) are the 2-regular \( \mathbb{F}_q \) conjugacy classes of \( D_{30} \) showing \( c = 8 \).

(2) \( n \equiv \pm 1 \mod 4 \)

\( T = \{1, 2, 4, 8\} \mod 15 \) and \( \{1\}, \{a, a^{-1}, a^2, a^{-2}, a^4, a^{-4}, a^7, a^{-7}\}, \{a^3, a^{-3}, a^6, a^{-6}\}, \{a^5, a^{-5}\} \) are the 2-regular \( \mathbb{F}_q \) conjugacy classes of \( D_{30} \) and hence, \( c = 4 \).

(3) \( n \equiv 2 \mod 4 \)

\( T = \{1, 4\} \mod 15 \) and \( \{1\}, \{a, a^{-1}, a^4, a^{-4}\}, \{a^2, a^{-2}, a^7, a^{-7}\}, \{a^3, a^{-3}\}, \{a^5, a^{-5}\}, \{a^6, a^{-6}\} \) are the 2-regular \( \mathbb{F}_q \) conjugacy classes of \( D_{30} \) giving \( c = 6 \).
Let $K$ be a finite extension of $\mathbb{F}_q$ containing a primitive root of order 15, say $\zeta$. For each $i$, $1 \leq i \leq 7$, the assignment
\[
a \mapsto \begin{bmatrix} \zeta^i & 0 \\ 0 & \zeta^{-i} \end{bmatrix}; \quad b \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\]
gives a well-defined group homomorphism $\theta_i : D_{30} \to GL(2, K)$ which can further be extended to a $K$-algebra homomorphism $f_i : K[D_{30}] \to M(2, K)$. For any $i, f_i$ is onto as $f_i(1), f_i(a), f_i(b)$ and $f_i(ab)$ are linearly independent over $K$. Thus if $M_i = \ker f_i$, then $K[D_{30}]/M_i \cong M(2, K)$ showing that $M_i$ is a maximal ideal of $K[D_{30}]$. Also, for any $i \neq j$, $1 \leq i, j \leq 7$, $M_i$ and $M_j$ are pairwise co-maximal as $x_i = a^2 - (\zeta^i + \zeta^{-i})a + 1 \in M_i$ but $x_i \notin M_j$.

Again the assignment $a \mapsto 1; b \mapsto 1$ gives a well-defined group homomorphism $\theta : D_{30} \to K^*$. Let $g : K[D_{30}] \to K$ be the corresponding $K$-algebra homomorphism. If $M = \ker g$, then it can be seen that $\{M, M_i \mid 1 \leq i \leq 7\}$ is a collection of pairwise co-maximal ideals of $K[D_{30}]$ and hence if $W = M \cap (\bigcap_{i=1}^{7} M_i)$, then by Chinese remainder theorem [2, Chapter 4, Theorem 2, pp. 110],
\[
(3.4) \quad \frac{K[D_{30}]}{W} \cong \frac{K[D_{30}]}{M} \oplus \left( \bigoplus_{i=1}^{7} \frac{K[D_{30}]}{M_i} \right) \cong K \oplus M(2, K)^7.
\]
This gives a surjective $K$-algebra homomorphism $\phi : K[D_{30}] \to K \oplus M(2, K)^7$ with $\ker \phi = W$. Now, $\phi(J(K[D_{30}])) \subseteq J(K \oplus M(2, K)^7) = (0)$ showing that $J(K[D_{30}]) \subseteq W$ and thus $\dim_K J(K[D_{30}]) \leq 1$. But $\text{char } K = 2 \mid |D_{30}|$. Hence $J(K[D_{30}]) \neq (0)$ so that $\dim_K J(K[D_{30}]) \geq 1$. Therefore, $J(K[D_{30}]) = W = \ker \phi$ and
\[
(3.5) \quad \frac{K[D_{30}]}{J(K[D_{30}])} \cong K \oplus M(2, K)^7
\]
By [3, Theorem 7.9], $J(K[D_{30}]) \cong K \otimes_{\mathbb{F}_2} J(\mathbb{F}_2[D_{30}])$ and hence $\dim_{\mathbb{F}_2} J(\mathbb{F}_2[D_{30}]) = \dim_K J(K[D_{30}]) = 1$.

As a particular case of equation (3.5), we have
\[
(3.6) \quad \frac{\mathbb{F}_2[D_{30}]}{J(\mathbb{F}_2[D_{30}])} \cong \mathbb{F}_2 \oplus M(2, \mathbb{F}_2)^7
\]
Now if we have $\frac{\mathbb{F}_2[D_{30}]}{J(\mathbb{F}_2[D_{30}])} \cong \bigoplus_{i=1}^{4} M(n_i, K_i) \cong \mathbb{F}_2 \oplus (\bigoplus_{i=1}^{3} M(n_i, K_i))$, then
\[
\frac{\mathbb{F}_2[D_{30}]}{J(\mathbb{F}_2[D_{30}])} \cong \mathbb{F}_2 \otimes_{\mathbb{F}_2} \frac{\mathbb{F}_2[D_{30}]}{J(\mathbb{F}_2[D_{30}])}
\]
The unit group of $\mathbb{F}_q[D_{30}]$ isomorphic to 

$$
\cong \mathbb{F}_{2^4} \otimes_{\mathbb{F}_2} (\mathbb{F}_2 \oplus (\oplus_{i=1}^3 M(n_i, K_i))) 
$$

(3.7)

By using (3.6) and (3.7), it can be seen that $n_i = 2$ and $\mathbb{F}_{2^4} \otimes_{\mathbb{F}_2} K_i \cong \mathbb{F}_{2^4}^{[K_i : \mathbb{F}_2]}$ for $i = 1, 2, 3$. Thus $y^{2^4} = y \forall y \in K_i$. If $K_i \cong \mathbb{F}_{2^4}$, then $k_i | 4$ and hence $k_i = 1, 2$ or $4$.

Now due to dimension constraints, $30 - 1 = 1 + 4 \left(\sum_{i=1}^3 [K_i : \mathbb{F}_2]\right)$ and hence $\sum_{i=1}^3 [K_i : \mathbb{F}_2] = 7$. The only possible choice for $([K_i : \mathbb{F}_2])_{i=1}^3$ is $(1, 2, 4)$ and hence

$$
\frac{\mathbb{F}_2[D_{30}]}{J(\mathbb{F}_2[D_{30}])} \cong \mathbb{F}_2 \oplus M(2, \mathbb{F}_2) \oplus M(2, \mathbb{F}_{2^2}) \oplus M(2, \mathbb{F}_{2^4})
$$

Let $\mathbb{F}_{2^n} \otimes_{\mathbb{F}_2} \mathbb{F}_{2^4} \cong \oplus_i \mathbb{F}_{2^{m_i}}$ (say). Then $m_i | \text{lcm}(n, 4)$ and considering the number of components $(c)$ in the decomposition, we have

$$
\cong \mathbb{F}_q \otimes_{\mathbb{F}_2} \frac{\mathbb{F}_2[D_{30}]}{J(\mathbb{F}_2[D_{30}])}
$$

$$
\cong \begin{cases} 
\mathbb{F}_q \oplus M(2, \mathbb{F}_q)^7 & \text{if } n \equiv \pm 1 \text{ mod } 4 \\
\mathbb{F}_q \oplus M(2, \mathbb{F}_q^2) \oplus M(2, \mathbb{F}_{2^2}) \oplus M(2, \mathbb{F}_{2^4}) & \text{if } n \equiv \pm 1 \text{ mod } 4 \\
\mathbb{F}_q \oplus M(2, \mathbb{F}_q)^3 \oplus M(2, \mathbb{F}_{2^2})^2 & \text{if } n \equiv \pm 2 \text{ mod } 4
\end{cases}
$$

and hence the theorem. $\square$

3.3. Char $\mathbb{F}_q = 3$.

Theorem 3.5. If $q = 3^n$, then,

$$
\mathbb{F}_q[D_{30}]/J(\mathbb{F}_q[D_{30}]) \cong \begin{cases} 
\mathbb{F}_q \oplus \mathbb{F}_q \oplus M(2, \mathbb{F}_{2^2}) & \text{if } n \text{ is odd} \\
\mathbb{F}_q \oplus \mathbb{F}_q \oplus M(2, \mathbb{F}_q^2) \oplus M(2, \mathbb{F}_{2^4}) & \text{if } n \text{ is even}
\end{cases}
$$

Proof. The 3 regular elements in $D_{30}$ are $1, a^3, a^{-3}, a^6, a^{-6}, b, ab, \ldots, a^{14}b$. Here, $m = 10$. We discuss $T$ and the 3-regular $\mathbb{F}_q$-conjugacy classes for the following cases:

1. $n \equiv 0 \text{ mod } 4 : T = \{1\}$ mod 10 and the 3-regular $\mathbb{F}_q$-conjugacy classes are $\{1, a^3, a^{-3}, a^6, a^{-6}, b, ab, \ldots, a^5b\}$ so that there are 4 simple components in the decomposition.
(2) $n \equiv \pm 1 \mod 4 : T = \{1, 3, 7, 9\} \mod 10$ and the $3$-regular $\mathbb{F}_q$-conjugacy classes are $\{1\}, \{a^3, a^{-3}, a^6, a^{-6}\}, \{b, ab, \cdots, a^{14}b\}$ as $(a^3)^7 = a^6$ and hence, $3$ simple components.

(3) $n \equiv 2 \mod 4 : T = \{1, 9\} \mod 10$ and the $3$-regular $\mathbb{F}_q$-conjugacy classes are $\{1\}, \{a^3, a^{-3}\}, \{a^6, a^{-6}\}, \{b, ab, \cdots, a^{14}b\}$ as in case 1 and so $4$ simple components in the decomposition.

We start with the case when $n$ is odd. Since $|\mathbb{F}_{q^2}| = 3^{2n} \equiv -1 \mod 5$, therefore, by Lemma 3.2, $\mathbb{F}_{q^2}$ contains $\sqrt{5}$. Let $\zeta_5$ be a primitive $5$th root of unity over $\mathbb{F}_q$.

Now, $\zeta_5^4 + \zeta_5^3 + \zeta_5^2 + \zeta_5 + 1 = 0 \Rightarrow (\zeta_5 + \zeta_5^{-1})^2 + (\zeta_5 + \zeta_5^{-1}) - 1 = 0$. Since the polynomial $f(x) = x^2 + x - 1$ has its roots $\left(\frac{-1 \pm \sqrt{5}}{2}\right)$ in $\mathbb{F}_{q^2} \setminus \mathbb{F}_q$, therefore $\alpha = \zeta_5 + \zeta_5^{-1} \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$. Hence, the assignment

$$a \mapsto \begin{bmatrix} -\alpha - 1 & 1 \\ -1 & 0 \end{bmatrix};$$

$$b \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

gives a well-defined group homomorphism $\eta_1 : D_{30} \to GL(2, \mathbb{F}_{q^2})$ which can be extended to an $\mathbb{F}_q$-algebra homomorphism

$$\eta_1^* : \mathbb{F}_q[D_{30}] \to M(2, \mathbb{F}_{q^2}).$$

$\eta_1^*$ is onto as $\eta_1^*(1), \eta_1^*(a), \eta_1^*(a^2), \eta_1^*(a^3), \eta_1^*(a^4), \eta_1^*(b), \eta_1^*(ab)$ and $\eta_1^*(a^2b)$ are linearly independent over $\mathbb{F}_q$ showing that $M(2, \mathbb{F}_{q^2})$ is a simple component in the Wedderburn decomposition of $\mathbb{F}_q[D_{30}] / J(\mathbb{F}_q[D_{30}])$. Since the assignments $a \mapsto 1, b \mapsto 1$ and $a \mapsto 1, b \mapsto -1$ give $2$ irreducible $\mathbb{F}_q$-representations of $D_{30}$ and by case 2 above, we have $3$ simple components for $n$ odd, hence for this case, we have

$$\frac{\mathbb{F}_q[D_{30}]}{J(\mathbb{F}_q[D_{30}])} \cong \mathbb{F}_q \oplus \mathbb{F}_q \oplus M(2, \mathbb{F}_{q^2}).$$

As a particular case, we have

$$\frac{\mathbb{F}_3[D_{30}]}{J(\mathbb{F}_3[D_{30}])} \cong \mathbb{F}_3 \oplus \mathbb{F}_3 \oplus M(2, \mathbb{F}_{3^2}).$$

Now, if $q = 3^n$, $n$ even, then,

$$\frac{\mathbb{F}_q[D_{30}]}{J(\mathbb{F}_q[D_{30}])} \cong \mathbb{F}_q \otimes_{\mathbb{F}_3} (\mathbb{F}_3 \oplus \mathbb{F}_3 \oplus M(2, \mathbb{F}_{3^2})).$$
The unit group of \( \mathbb{F}_q[D_{30}] \) 

\[
\cong \mathbb{F}_q \oplus \mathbb{F}_q \oplus M(2, \mathbb{F}_q) \oplus \mathbb{F}_q[3]
\]

\[
\cong \mathbb{F}_q \oplus \mathbb{F}_q \oplus M(2, \mathbb{F}_q) \oplus M(2, \mathbb{F}_q)
\]

by Lemma 3.1

and hence the result. \( \square \)

### 3.4. \( \text{Char } \mathbb{F}_q = 5 \).

**Theorem 3.6.** If \( q = 5^n \), then,

\[
\mathbb{F}_q[D_{30}]/J(\mathbb{F}_q[D_{30}]) \cong \mathbb{F}_q \oplus \mathbb{F}_q \oplus M(2, \mathbb{F}_q)
\]

**Proof.** The assignments \( a \mapsto 1, b \mapsto 1 \) and \( a \mapsto 1, b \mapsto -1 \) give two distinct linear \( \mathbb{F}_q \)-representations of \( D_{30} \). Also, the assignment

\[
a \mapsto \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix};
\]

\[
b \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 4 \end{bmatrix}
\]

gives a well-defined group homomorphism \( \eta_2 : D_{30} \to GL(2, \mathbb{F}_q) \) which can be extended to an \( \mathbb{F}_q \)-algebra homomorphism

\[
\eta_2^* : \mathbb{F}_q[D_{30}] \to M(2, \mathbb{F}_q).
\]

\( \eta_2^* \) is onto as \( \eta_2^*(1), \eta_2^*(a), \eta_2^*(b) \) and \( \eta_2^*(ab) \) is linearly independent over \( \mathbb{F}_q \) showing that \( M(2, \mathbb{F}_q) \) is a simple component in the Wedderburn decomposition of \( \mathbb{F}_q[D_{30}]/J(\mathbb{F}_q[D_{30}]) \). Now, \( m = 6 \) and the 5 regular elements in \( D_{30} \) are 1, \( a^5 \), \( a^{-5} \), \( b \), \( ab \), \ldots, \( a^{14}b \). For the following cases, \( T \) is as follows

1. \( n \) odd : \( T = \{1, 5\} \mod 6 \)
2. \( n \) even : \( T = \{1\} \mod 6 \)

In both the cases, the 5-regular \( \mathbb{F}_q \)-conjugacy classes are \{1\}, \{\( a^5, a^{-5} \}\}, \{b, ab, \ldots, a^{14}b\} and hence, 3 simple components in the decomposition. Thus, by all the observations above, we conclude,

\[
(3.8) \quad \frac{\mathbb{F}_q[D_{30}]}{J(\mathbb{F}_q[D_{30}])} \cong \mathbb{F}_q \oplus \mathbb{F}_q \oplus M(2, \mathbb{F}_q)
\]

and hence the result. \( \square \)
4. Proof of the main result. For a group $G$ and a field $F$, if $J(F[G])^n = (0)$, then $V_1 = 1 + J(F[G])$ is a nilpotent group of class at most $n - 1$. A group $G$ is said to be $p$-solvable if the composition factors of $G$ are either $p$-groups or $p'$-groups. By [5, Chapter III, Proposition 1.9, pp. 110], if $G$ is a $p$-solvable group of order $np^q$ where $(n, p) = 1$ and $F$ is a field of characteristic $p$, then $J(F[G])^{p^q} = (0)$. Since a finite solvable group is $p$-solvable for any prime $p$, therefore the result follows for solvable groups and in particular for $D_{30}$. We shall use this result to compute the nilpotency index of $J(F_q(D_{30}))$ and then to compute the nilpotency class of the group $V_1 = 1 + J(F_q(D_{30}))$.

Proof. When $p > 5$, the result follows from the Wedderburn decomposition as in Theorem 3.3. Since $U(F_q[D_{30}]) \cong V_1 \times U(F_q[D_{30}]/J(F_q[D_{30}]))$, the main result for the cases when $p = 2, 3$ and 5 follows from the Wedderburn decomposition of $F_q[D_{30}]/J(F_q[D_{30}])$ obtained in Theorem 3.4, 3.5 and 3.6 respectively in Section 3 and the details about $V_1$ given below.

1. If $p = 2$, then for any $y \in F_q[D_{30}]$,

\[
(1 + yD_{30})^2 = 1 + (yD_{30})^2 = 1 + y^2D_{30}^2 = 1 + 30y^2D_{30} = 1.
\]

Thus $1 + yD_{30}$ is invertible for any $y \in F_q[D_{30}]$ and hence $D_{30} \in J(F_q[D_{30}])$. Since $\dim_{F_q} J(F_q[D_{30}]) = 1$, $J(F_q[D_{30}]) = F_qD_{30} \subseteq Z(F_q[D_{30}])$. Thus $V_1 \cong C_2^n$ and $U(F_q[G]) \cong V_1 \times U(F_q[G])$.

2. If $p = 3$, then we have $J(F_q[D_{30}])^3 = (0)$ and hence nilpotency class of $V_1 \leq 2$ and $x^3 = 1 \forall x \in V_1$.

Let $x = 1 - a^5$, $y = b(1 - a^5)$. If $\gamma \in F_q[D_{30}]$, then $\gamma = \alpha + \beta b$, for some $\alpha, \beta \in F_q[a]$. We claim that $((1 - a^5)\gamma)^3 = 0$.

Now,

\[
((1 - a^5)\gamma)^3 = ((1 - a^5)(\alpha + \beta b))^3 = (X + Y)^3, \text{ where } X = (1 - a^5)\alpha \text{ and } Y = (1 - a^5)\beta b
\]
\[
= X^3 + X^2Y + XY^2 + YX^2 + YXY + Y^2X + Y^3
\]
\[
= (1 - a^5)^3\alpha^3 + a^2\beta(1 - a^5)^3b + \alpha\beta(1 - a^5)^2b(1 - a^5)\alpha
\]
\[
+ \alpha\beta(1 - a^5)^2b(1 - a^5)\beta b + \beta(1 - a^5)b(1 - a^5)\beta b(1 - a^5)\alpha
\]

Thus, $1 + yD_{30}$ is a field of characteristic $p$. If $\gamma \in F_q[D_{30}]$, then $\gamma = \alpha + \beta b$, for some $\alpha, \beta \in F_q[a]$. We claim that $((1 - a^5)\gamma)^3 = 0$. Now,

\[
((1 - a^5)\gamma)^3
\]

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\[ + \beta(1-a^5)b(1-a^5)\beta b(1-a^5)\beta b \]

Since $ba = a^{-1}b$, it is clear that for any $\mu \in \mathbb{F}_q[(a)]$, $b\mu = \mu^*b$ for some $\mu^* \in \mathbb{F}_q[(a)]$. Using this in the above, we get

\[
((1-a^5)\gamma)^3 = (1-a^5)^3\alpha^3 + \alpha^2\beta(1-a^5)^3b + \alpha\beta(1-a^5)^2(1-a^{10})b\alpha \\
+ \alpha\beta(1-a^5)(1-a^{10})b\beta b + \beta(1-a^5)(1-a^{10})^2b\alpha^2 \\
+ \beta(1-a^5)(1-a^{10})^2b\alpha\beta b + \beta\beta^*\alpha(1-a^5)(1-a^{10})(1-a^5) \\
+ \beta\beta^*\beta(1-a^5)(1-a^{10})(1-a^5)b \\
\]

As $(1-a^{10}) = (1-a^5)(1+a^5)$ and $(1-a^5)^3 = 0$, it follows that \((1-a^5)^3 = 0 \forall \gamma \in \mathbb{F}_q[D_{30}]\). This gives

\[
(1 + (1-a^5)\gamma)^3 = 1 \forall \gamma \in \mathbb{F}_q[D_{30}] \\
\Rightarrow 1 + (1-a^5)\gamma \text{ is invertible } \forall \gamma \in \mathbb{F}_q[D_{30}] \\
\Rightarrow x = 1 + (1-a^5) \in J(\mathbb{F}_q[D_{30}]) \\
\]

Thus, $x, y \in J(\mathbb{F}_q[D_{30}])$ but $xy \neq yx$ showing that $V_1$ is not abelian and hence it is a nilpotent class 2 group of exponent 3.

3. If $p = 5$, then $J(\mathbb{F}_q[D_{30}])^5 = (0)$ and hence, $V_1$ has exponent 5 and nilpotency class of $V_1 \leq 4$. By similar arguments as in the previous case, we find that $a^3 - 1 = 4 + a^3 \in J(\mathbb{F}_q[D_{30}])$ and therefore $4b + a^9b, 4a^2 + a^8, 4ab + a^{10}b \in J(\mathbb{F}_q[D_{30}])$. Now, if we take $x = a^3$, $y = 1 + 4b + a^9b$, $z = 1 + 4a^2 + a^8$ and $w = 1 + 4ab + a^{10}b \in V_1$, then

$A = (x, y) = 4 + a^3 + 4a^6 + 2a^9 + 6b + 3a^9b + a^{12}b,$

$B = (z, A) = 1 + b + 3ab + 3a^2b + a^3b + a^4b + a^5b + a^6b + 4a^7b + 4a^8b + a^9b + 2a^{10}b + 2a^{11}b + a^{12}b,$

$C = (w, B) = 1 + a + 4a^2 + a^4 + 4a^5 + a^7 + 4a^8 + a^{10} + 4a^{11} + a^{13} + 4a^{14} \neq 1$ and hence, $V_1$ is a nilpotent group of class 4. $\Box$

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