LIE SOLVABLE RINGS

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ABSTRACT. Let \( \mathcal{L}(R) \) denote the associated Lie ring of an associative ring \( R \) with identity \( 1 \neq 0 \) under the Lie multiplication \( [x, y] = xy - yx \) with \( x, y \in R \). Further, suppose that the Lie ring \( \mathcal{L}(R) \) is solvable of length \( n \). It has been proved that if 3 is invertible in \( R \), then the ideal \( J \) of \( R \) generated by all elements \( [[x_1, x_2], [x_3, x_4]], x_1, x_2, x_3, x_4, x_5 \in R \), is nilpotent of index at most \( \frac{1}{3} \left( 19 \cdot 10^{n-3} - 1 \right) \) for \( n \geq 3 \). Also, if 2 and 3 are both invertible in \( R \), then the ideal \( I \) of \( R \) generated by all elements \( [x, y], x, y \in R \), is a nil ideal of \( R \). Some applications to Lie solvable group rings are also given.

Let \( R \) be any associative ring with identity \( 1 \neq 0 \). We can induce the Lie structure on \( R \) by defining the Lie product \( [x, y] = xy - yx \) for \( x, y \in R \). The Lie ring thus obtained is called the associated Lie ring of \( R \) and is denoted by \( \mathcal{L}(R) \). Jennings [1] proved that if \( \mathcal{L}(R) \) is nilpotent then the associative ideal of \( R \) generated by all elements \( [[x, y], z], x, y, z \in R \), is a nilpotent ideal of \( R \) and the ideal generated by all \( [x, y], x, y \in R \), is nil. In this paper we study the case when \( \mathcal{L}(R) \) is solvable.

1. Lie identities and Lie ideals. Let \( x_1, x_2, \ldots, x_n \in R \); then the left normed commutators are defined by \( [x_1, x_2] = x_1x_2 - x_2x_1 \) and, inductively,

\[
[x_1, x_2, \ldots, x_n] = [[[x_1, x_2], [x_3, \ldots, x_{n-1}]], x_n].
\]

We shall repeatedly use the following well-known identities, which are easy to prove.

**Lemma 1.1** For \( x, y, z \in R \), the following identities are true:
(i) \( [x, y] = -[y, x] \).
(ii) \( [x, y, z] + [y, z, x] + [z, x, y] = 0 \) (Jacobi identity).
(iii) \( [xy, z] = x[y, z] + [x, z]y \).
(iv) \( [x, yz] = y[x, z] + [x, y]z \).

For any two subsets \( A \) and \( B \) of \( R \), by \( [A, B] \) we shall denote the additive subgroup of \( R \) generated by all elements \( [a, b] \) with \( a \in A \) and \( b \in B \). A Lie ideal of \( R \) means an ideal of the Lie ring \( \mathcal{L}(R) \). Thus, \( U \) is a Lie ideal if \( U \) is an additive subgroup of \( R \) and \( [a, b] \in U \) for \( a \in U \) and \( b \in R \). It is easy to see that \( [U, V] \) is a Lie ideal if \( U \) and \( V \) are Lie ideals.

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Let $U$ be any Lie ideal of $R$. In view of the identity $ur = [u, r] + ru$, $u \in U$, $r \in R$, the right and left ideals of $R$, generated by $U$, are identical. In particular, $RU = UR$ is the two-sided ideal generated by $U$. Also, $U^n$ consists of finite sums of $m$-fold products $u_1 u_2 \cdots u_m$ with $u_1, u_2, \ldots, u_m \in U$. Therefore, $(UR)^n = U^nR$ for any Lie ideal $U$ and for any positive integer $n$.

The derived chain of any Lie ideal $V$ is given by

$$U = \delta^{(0)}(U) \supseteq \delta^{(1)}(U) \supseteq \delta^{(2)}(U) \supseteq \cdots \supseteq \delta^{(n)}(U) \supseteq \cdots$$

where $\delta^{(n+1)}(U) = [\delta^{(n)}(U), \delta^{(n)}(U)]$, $n \geq 0$. We say that $\mathcal{L}(R)$ is solvable of length $n$ if $\delta^{(n)}(\mathcal{L}(R)) = (0)$, $n$ least.

Further, the lower central chain of $V$ is defined by

$$U = \gamma_1(U) \supseteq \gamma_2(U) \supseteq \gamma_3(U) \supseteq \cdots \supseteq \gamma_n(U) \supseteq \cdots$$

where $\gamma_{n+1}(U) = [\gamma_n(U), U]$, $n \geq 1$. The Lie ring $\mathcal{L}(R)$ is nilpotent of class $n$ if $\gamma_{n+1}(\mathcal{L}(R)) = (0)$, $n$ least.

We proceed with a sequence of lemmas needed for our further work. These lemmas are also of independent interest.

**Lemma 1.2.** Let $U$ be a Lie ideal of $R$; then

(i) $[U^n, \mathcal{L}(R)] \subseteq [U, \mathcal{L}(R)] \subseteq U$.

(ii) $[\delta^{(1)}(U)R, \delta^{(1)}(\mathcal{L}(R))R] \subseteq U$.

**Proof.** (i) follows by induction on $m$ and the identity $[u_1 u_2, r] = [u_1, u_2 r] + [u_2, u_1 r]$.

To prove (ii), for $u_1, u_2, u_3, u_4 \in U$ and $r, s \in R$, we have the identity

$$[[u_1, u_2 r], [u_3, u_4 s]] - [u_2 [u_1, r], u_4 [u_3, s]] - [u_2 [u_1, r], [u_3, u_4 s]] + [u_4 [u_3, s], [u_1, u_2 r]].$$

This can easily be obtained by expanding the first term on the right and using Lemma 1.1.

Now the right side belongs to $U$ by (i). The lemma follows easily.

**Lemma 1.3.** Let $U$ be a Lie ideal of $R$; then

(i) $(\delta^{(1)}(U))^2 \cdot \delta^{(1)}(\mathcal{L}(R)) \subseteq U + [\delta^{(1)}(U), \mathcal{L}(R)]R$, and

(ii) $(\delta^{(1)}(U))^2 \cdot \delta^{(2)}(\mathcal{L}(R)) \subseteq \delta^{(1)}(U) + [\delta^{(1)}(U), \mathcal{L}(R)]R$.

**Proof.** Let $v_1, v_2$ be two-fold commutators in the elements of $U$ and let $r_1, r_2 \in R$. Expanding the first term on the right and using Lemma 1.1, we get the identity

$$v_1 v_2 [r_1, r_2] = [v_1 r_1, v_2 r_2] - [v_1, v_2 r_2] r_1 - v_1 [r_1, v_2] r_2.$$

By Lemma 1.2(ii), $[v_1 r_1, v_2 r_2] \subseteq U$, and the other two terms clearly belong to $[\delta^{(1)}(U), \mathcal{L}(R)]R$. This proves (i).

Now let $w_1, w_2, w_3, w_4$ be two-fold commutators in the elements of $U$, and let $s_1, s_2$ be two-fold commutators in the elements of $\mathcal{L}(R) = R$. Then the following identity can be obtained by using Lemma 1.1 and expanding the first term on the right:

$$w_1 w_2 w_3 w_4 [s_1, s_2] = [w_1 w_2 s_1, w_3 w_4 s_2] - w_1 w_2 w_3 [s_1, w_4] s_2 - w_1 w_2 [s_1, w_3] w_4 s_2 + w_1 w_2 w_3 w_4 [s_1, w_2] w_4 s_2.$$
The first term on the right, by (i), easily belongs to $\delta^{(1)}(U) + [\delta^{(1)}(U), \mathcal{L}(R)]R$, and all other terms on the right are clearly in $[\delta^{(1)}(U), \mathcal{L}(R)]R$ because it is a two-sided ideal. Thus we get (ii).

This leads to our next lemma. Let $J$ denote the ideal of $R$ generated by all elements $[[x_1, x_2], [x_3, x_4], x_5]$ with $x_1, \ldots, x_5 \in R$. Clearly,

$$J = [\delta^{(1)}(\mathcal{L}(R)), \delta^{(1)}(\mathcal{L}(R)), \mathcal{L}(R)]R = [\delta^{(2)}(\mathcal{L}(R)), \mathcal{L}(R)]R.$$

**Lemma 1.4.** For any Lie ideal $U$ of $R$, we have

$$(\delta^{(1)}(U))^4 \cdot J \subseteq [\delta^{(1)}(U), \mathcal{L}(R)]R.$$

**Proof.** The left side is a finite sum of elements of the type $ab, c]r$, with $a \in (\delta^{(1)}(U))^4$, $b \in \delta^{(2)}(\mathcal{L}(R))$, $c \in \mathcal{L}(R)$, and $r \in R$. But

$$a[b, c] = [ab, c] - [a, c]b.$$

By Lemma 1.3(ii), $ab \in \delta^{(1)}(U) + [\delta^{(1)}(U), \mathcal{L}(R)]R$. Hence,

$$[ab, c] \in [\delta^{(1)}(U), \mathcal{L}(R)]R.$$

Also, by Lemma 1.2(i),

$$a[b, c]r \in [\delta^{(1)}(U), \mathcal{L}(R)]R \quad \text{and the lemma is proved.}$$

**Lemma 1.5.** For any Lie ideal $U$ of $R$, we have

$$\delta^{(1)}(U)[\delta^{(1)}(U), \mathcal{L}(R)] \subseteq \gamma_3(U)R.$$

**Proof.** Let $u_1, u_2, u_3, u_4 \in U$ and $r \in R$. Then the following identity gives the result:

$$[u_1, u_2][u_3, u_4, r] = [[u_3, u_4], [u_1, u_2]r] - u_2[[u_3, u_4], [u_1, r]] - [[u_3, u_4], [u_1, u_2]]r - [u_3, u_4, u_2][u_1, r].$$

**Corollary 1.6.** Let $U$ be a Lie ideal of $R$; then

(i) $[\delta^{(1)}(U), \mathcal{L}(R)]^2 \subseteq \gamma_3(U)R$, and

(ii) $J^2 \subseteq \gamma_3(\delta^{(1)}(\mathcal{L}(R)))R$.

**Proof.** (i) follows by Lemma 1.5 since $[\delta^{(1)}(U), \mathcal{L}(R)] \subseteq \delta^{(1)}(U)$.

(ii) follows from (i) if we put $U = \delta^{(1)}(\mathcal{L}(R))$ and observe that

$$J = [\delta^{(2)}(\mathcal{L}(R)), \mathcal{L}(R)]R = R[\delta^{(2)}(\mathcal{L}(R)), \mathcal{L}(R)].$$

The next lemma is crucial to our further work. Its proof also requires some computations.

**Lemma 1.7.** Let $U$ be a Lie ideal of a ring $R$ in which 3 is invertible. Then

$$\gamma_3(U)^2 \subseteq \delta^{(2)}(U)R.$$

**Proof.** It is enough to show that $[u_1, u_2, u_3]u_4, u_5, u_6] \in \delta^{(2)}(U)R$ for all $u_1, \ldots, u_6 \in U$. To do this, we proceed as follows. Let

$$a = [u_1, u_2, u_3][u_4, u_5, u_6] + [u_1, u_2, u_6][u_5, u_4, u_3].$$
Observe that the second term can be obtained from the first by interchanging \(u_5\) (the last entry of the first bracket) and \(u_6\) (the first entry of the second bracket). Expanding \([[u_4, u_6u_5, u_3], [u_1, u_2]]\) properly, we can easily get
\[
a = [[u_4, u_6u_5, u_3], [u_1, u_2]] + [[u_1, u_2, u_3], [u_6, u_4, u_3]]
- u_6[[u_4, u_5, u_3], [u_1, u_2]] - [[u_4, u_6, u_3], [u_1, u_2]]u_5
- [u_6, u_3][[u_4, u_5], [u_1, u_2]] - [[u_6, u_3], [u_1, u_2]][u_4, u_5]
- [u_4, u_6][[u_5, u_3], [u_1, u_2]] - [[u_4, u_6], [u_1, u_2]][u_5, u_3].
\]
Certainly, \(a \in \delta^{(2)}(U)R\).

In an exactly similar manner,
\[
b = [u_1, u_2, u_4][u_6, u_5, u_3] + [u_1, u_2, u_6][u_4, u_5, u_3] \in \delta^{(2)}(U)R.
\]

We now turn to the case when the last entry of the first bracket and the last entry of the second bracket are interchanged.

Expanding \([[u_6, u_4], [[u_1, u_2], u_3u_5]]\) and rearranging terms, we get
\[
c = [u_1, u_2, u_3][u_6, u_4, u_5] + [u_1, u_2, u_3][u_6, u_4, u_3]
- [u_6, u_4][[u_1, u_2, u_3], u_5] - [[u_6, u_4], [u_1, u_2, u_3]]u_5.
\]
This shows that \(c \in \delta^{(2)}(U)R\).

Arguments, as in the case of \(c\), will also give
\[
d = [u_1, u_2, u_3][u_6, u_5, u_4] + [u_1, u_2, u_4][u_6, u_5, u_3] \in \delta^{(2)}(U)R
\]
and
\[
e = [u_1, u_2, u_3][u_4, u_5, u_6] + [u_1, u_2, u_6][u_4, u_5, u_3] \in \delta^{(2)}(U)R.
\]
Finally, by using Lemma 1.1(ii) and rearranging terms, we get
\[
3[u_1, u_2, u_3][u_4, u_5, u_6] = a - b - c + d + 2e \in \delta^{(2)}(U)R.
\]
Since 3 is invertible in \(R\),
\[
[u_1, u_2, u_3][u_4, u_5, u_6] \in \delta^{(2)}(U)R.
\]
This completes the proof.

**Corollary 1.8.** If 3 is invertible in \(R\), then \(J^4 \subseteq \delta^{(3)}(\mathcal{L}(R))R\).

**Proof.** \(J^2 \subseteq \gamma_3(\delta^{(1)}(\mathcal{L}(R)))R\) by Corollary 1.6. Therefore,
\[
J^4 \subseteq (\gamma_3(\delta^{(1)}(\mathcal{L}(R))))^2R
\subseteq \delta^{(2)}(\delta^{(1)}(\mathcal{L}(R)))R \quad (\text{by Lemma 1.7})
= \delta^{(3)}(\mathcal{L}(R))R.
\]

The next lemma does not assume that 3 is invertible in \(R\) and has a much simpler proof than Lemma 1.7.

**Lemma 1.9.** Let \(U\) be a Lie ideal of \(R\) such that \(U\) is also a subring of \(R\); then
\((\gamma_3(U))^2 \subseteq \delta^{(2)}(U)R\).
PROOF. Expanding the first term on the right, we have

\[ [u_1, u_2, u_3][u_4, u_5, u_6] = [[[u_4, u_5], [u_1, u_2]u_6, u_3]] \\
+ [[[u_1, u_2, u_3], [u_4, u_5]]u_6 + [u_1, u_2][u_6, u_3], [u_4, u_5]] \\
+ [[u_1, u_2], [u_4, u_5]][u_6, u_3] \]

for all \( u_1, \ldots, u_6 \in U \). Clearly the right side belongs to \( \delta^2(U)R \), as desired.

COROLLARY 1.10. For any ring \( R \), \( (\gamma_3(\mathcal{L}(R)))^2 \subseteq \delta^2(\mathcal{L}(R))R \).

2. Main results. In this section we prove our main theorems.

THEOREM 2.1. Let \( R \) be a ring in which \( 3 \) is invertible, and let its associated Lie ring \( \mathcal{L}(R) \) be solvable of length \( n \geq 3 \). Then the ideal \( J \) of \( R \), generated by all elements \( [[x_1, x_2], [x_3, x_4], x_5] \) with \( x_1,\ldots,x_5 \) in \( R \), is nilpotent of index at most \( \frac{1}{3}(19 \cdot 10^{n-3} - 1) \).

PROOF. If \( \mathcal{L}(R) \) is solvable of length \( n = 3 \), then \( \delta^3(\mathcal{L}(R)) = (0) \). By Corollary 1.8, \( J^4 = (0) \). So the theorem is true for \( n = 3 \).

We assume that \( n \geq 4 \). Now for any Lie ideal \( U \) of \( R \), using Lemmas 1.4 and 1.5, we get

\[ \delta^3(U)(\delta^4(U))^4J \subseteq \gamma_3(U)R. \]

Thus, by Lemma 1.7,

\[ \left\{ (\delta^3(U))^5J \right\}^2 \subseteq (\gamma_3(U))^2R \subseteq \delta^2(U)R. \]

Putting \( U = \delta^{m-2}(\mathcal{L}(R)) \), we get

\[ \left\{ (\delta^{m-1}(\mathcal{L}(R)))^5J \right\}^2 \subseteq \delta^m(\mathcal{L}(R))R \]

for all \( m \geq 4 \).

Thus, for \( m = 4 \), we have

\[ \left\{ (\delta^3(\mathcal{L}(R)))^5J \right\}^2 \subseteq \delta^4(\mathcal{L}(R))R \]

and, using Corollary 1.8,

\[ \left\{ (J^4)^5J \right\}^2 = J^{1 + 2 \cdot 10^2} \subseteq \delta^4(\mathcal{L}(R))R. \]

We claim that, by induction on \( m \),

\[ J^{2(1 + 10 + 10^2 + \cdots + 10^{n-4} + 2 \cdot 10^{n-3})} \subseteq \delta^m(\mathcal{L}(R))R \]

for all \( m \geq 4 \).

Assume this is true for \( m \) and use \( ((\delta^m(\mathcal{L}(R)))^5J)^2 \subseteq \delta^{m+1}(\mathcal{L}(R))R \) to prove it for \( m + 1 \). Thus,

\[ J^N \subseteq \delta^n(\mathcal{L}(R))R = (0), \]

where

\[ N = 2\left(1 + 10 + 10^2 + \cdots + 10^{n-4} + 2 \cdot 10^{n-3}\right) = \frac{1}{3}(19 \cdot 10^{n-3} - 1), \]

as desired.
Next, we prove that the ideal $I$ of $R$, generated by all elements $[x, y], x, y \in R$, is a nil ideal if 2 and 3 are both invertible in $R$. First, we need the following

**Lemma 2.2.** Let $R$ be a ring in which 2 is invertible. Then

(i) $[[x_1, x_2], [x_3, x_4]]^3 \in J$ for all $x_1, \ldots, x_4 \in R$,

(ii) $[x, y, z]^{10} \in J$ for all $x, y, z \in R$, and

(iii) $[x, y]^{21} \in J$ for all $x, y \in R$.

**Proof.** Expanding the first term on the right, we get

$$2[[x_1, x_2], [x_3, x_4]]^2 = [[x_1, x_2]^2, [x_3, x_4], [x_3, x_4]]$$

$$- [x_1, x_2][[x_1, x_2], [x_3, x_4], [x_3, x_4]]$$

$$- [[x_1, x_2], [x_3, x_4], [x_3, x_4]][x_1, x_2].$$

Thus,

$$2[[x_1, x_2], [x_3, x_4]]^2 = [[x_1, x_2]^2, [x_3, x_4], [x_3, x_4]] \pmod{J}.$$ 

Also,

$$2[[x_1, x_2]^2, [x_3, x_4], [x_3, x_4]] [[x_1, x_2], [x_3, x_4]]$$

$$= [[[x_1, x_2]^2, [x_3, x_4], [x_3, x_4]], [x_1, x_2], [x_3, x_4]]$$

$$+ [[[x_1, x_2]^2, [x_3, x_4], [x_3, x_4], [x_1, x_2], [x_3, x_4]]]$$

$$+ [[[x_1, x_2]^2, [x_3, x_4], [x_3, x_4], [x_1, x_2], [x_3, x_4]]$$

$$= 0 \pmod{J}.$$

Combining, we get $4[[x_1, x_2], [x_3, x_4]]^3 \in J$, and, since 2 is invertible, we get (i).

To prove (ii), observe the identity

$$[x, y, z]^2 = [[[x, y], [x, y]z, z]] + [[x, y, z], [x, y]]z,$$

and use (i), keeping in view that

$$r[[x_1, x_2], [x_3, x_4]] = [[x_1, x_2], [x_3, x_4]]r - [[x_1, x_2], [x_3, x_4], r].$$

Similarly, to prove (iii) it is enough to use (i) and see that

$$[x, y]^3 = [[[xy, y], [yx, x]] + [[[y, x], [xy, y]], x] + [[[xy, x], [x, y]]y.$$

Note that powers given in Lemma 2.2 are not the best possible; the purpose is served by proving that some power in each case belongs to $J$.

**Theorem 2.3.** Let $R$ be a ring in which both 2 and 3 are invertible, and let $I_0$ be the ideal of $R$ generated by all elements $[x, y, z], x, y, z \in R$. If the associated Lie ring $\mathcal{L}(R)$ is solvable, then $I_0$ is a nil ideal.

**Proof.** Clearly, $I_0 = \gamma_5(\mathcal{L}(R))R = R\gamma_5(\mathcal{L}(R))$. By Corollary 1.10, $I_0^2 = (\gamma_5(\mathcal{L}(R)))^2R \subseteq \delta(\mathcal{L}(R))R$. Now suppose $\mathcal{L}(R)$ is solvable of length $n$, so $\delta^{(n)}(\mathcal{L}(R)) = (0)$. If $n = 1, I_0 = (0)$. If $n = 2, I_0^2 = (0)$. So assume $n \geq 3$. 


Let $\alpha \in I_0$; then $\alpha^2 \in I_0^2 \subseteq \delta^{(2)}(\mathcal{L}(R))R$ and, hence,
\[
\alpha^2 = \sum_{i=1}^{m} ([x_i, y_i], [u_i, v_i])r_i = \sum_{i=1}^{m} \alpha_ir_i,
\]
where $\alpha_i = ([x_i, y_i], [u_i, v_i])$ for $i = 1, 2, \ldots, m$. By Lemma 2.2(i) each $\alpha_i \in J$. Also, $[\alpha_i, r] \in J$ for every $r \in R$. Further, $\alpha_i\beta\alpha_i = \alpha_i^2\beta - \alpha_i[\alpha_i, \beta]$ for any $\beta \in R$. Thus,
\[
\alpha_i\beta\alpha_i = \alpha_i^2\beta \quad (\text{mod } J)
\]
and, similarly,
\[
\alpha_i\beta\alpha_i\theta\alpha_i = \alpha_i^2\beta\theta \quad (\text{mod } J) = 0 \quad (\text{mod } J)
\]
as $\alpha_i^2 \in J$.

The above arguments, applied to $(\alpha^2)^k = (\sum_{i=1}^{m} \alpha_ir_i)^k$ for $k \geq 2m + 1$, immediately give that $(\alpha^2)^k \in J$ for $k \geq 2m + 1$. But, by Theorem 2.1, $J$ is nilpotent, hence $((\alpha^2)^k)^N = 0$ for suitable $N$. This proves that $\alpha$ is nilpotent for every $\alpha \in I_0$. That is, $I_0$ is a nil ideal.

In fact, we are able to obtain a much stronger result.

**Theorem 2.4.** Let $R$ be a ring in which both 2 and 3 are invertible, and let $I$ be the ideal of $R$ generated by all elements $[x, y], x, y \in R$. If the associated Lie ring $\mathcal{L}(R)$ is solvable, then $I$ is a nil ideal.

**Proof.** Let $\gamma_2(\mathcal{L}(R)) = \delta^{(1)}(\mathcal{L}(R))R$. Let $\mathcal{L}(R)$ be solvable of length $n$; then $\delta^{(n)}(\mathcal{L}(R)) = (0)$. Therefore, if $n = 1, I = (0)$.

Let $\alpha \in I$; then
\[
\alpha = \sum_{i=1}^{m} [x_i, y_i]r_i = \sum_{i=1}^{m} \alpha_ir_i,
\]
where $\alpha_i = [x_i, y_i]$. By Lemma 2.2(iii), $\alpha_i^{21} = [x_i, y_i]^{21}$ always belongs to $J$.

Now,
\[
[x, y]r[x, y] = [x, y]^2r - [x, y][x, y, r] \equiv [x, y]^2r \quad (\text{mod } I_0).
\]
If we take $\alpha^k = (\sum_{i=1}^{m} \alpha_ir_i)^k$ for $k \geq 20m + 1$, then $\alpha^k$ will be a finite sum of $k$-fold products of elements from $\{\alpha_1r_1, \alpha_2r_2, \ldots, \alpha_mr_m\}$, and in each $k$-fold product some $\alpha_i$ will be repeated at least 21 times. Collecting repeatedly these factors, by the above process, the $k$-fold products will be congruent to $\alpha_i^{21}r \mod I_0$. Since $\alpha_i^{21} \in J$, this implies that $\alpha^k = \lambda + \mu$ with $\lambda \in J$ and $\mu \in I_0$. But, by Theorem 2.3, $I_0$ is a nil ideal, so $\mu' = 0$ for some $l$. This gives $(\alpha^k)' = \alpha_i^{2l} \in J$. Now use the nilpotency of $J$ to get $(\alpha^{k})^N = 0$. Thus, $\alpha$ is nilpotent and $I$ is a nil ideal.

3. Applications to group rings. Lie solvable group rings were studied by Passi, Passman and Sehgal in [2]. Let $K[G]$ denote the group ring of the group $G$ over the field $K$ with $\text{Char } K = p \geq 0, p \neq 2$. If $p > 0$ we say that a group $G$ is $p$-Abelian if the commutator subgroup $G'$ is a finite $p$-group. For convenience, 0-Abelian will mean Abelian. It was proved in [2] that if $\text{Char } K = p \neq 2$, then the associated Lie algebra $\mathcal{L}(K[G])$ of $K[G]$ is solvable if and only if $G$ is $p$-Abelian. Using Theorem 2.4 we get an alternative proof in characteristic 0 as follows.
Suppose $\text{Char } K = 0$ and $\mathcal{L}(K[G])$ is solvable. Then, by Theorem 2.4, $I = \gamma_2(\mathcal{L}(K[G])) \cdot K[G]$ is a nil ideal of $K[G]$. So, by [3, Theorem 2.3.4, p. 47], $I = (0)$. This gives $\gamma_2(\mathcal{L}(K[G])) = (0)$, i.e., $K[G]$ is commutative. Hence, $G$ is Abelian. Thus, $\mathcal{L}(K[G])$ is solvable if and only if $G$ is Abelian.

Also in $\text{Char } K = p > 0$, $p \neq 2, 3$, we have an advantage. Suppose $\mathcal{L}(K[G])$ is solvable. Then $I = [K[G], K[G]]K[G] = \omega(K[G']) \cdot K[G]$ is a nil ideal of $K[G]$ by Theorem 2.4. Hence, $\omega(K[G']) \cdot K[G]$ is contained in the Jacobson radical $J(K[G])$ of $K[G]$. By [3, Lemma 10.1.13, p. 415], $G'$ is a $p$-group. Also, $\mathcal{L}(K[G])$ is solvable implies $K[G]$ satisfies a polynomial identity. By [3, Theorem 5.2.14, p. 189], $[G : \Delta(G)] < \infty$ and $\Delta(G)$ is finite, where $\Delta(G)$ is the FC-subgroup of $G$. Now $G'/G' \cap \Delta(G)$ is a finite $p$-group, $G' \cap \Delta(G)/\Delta(G)$ is Abelian, and $\Delta(G)$ is a finite $p$-group implies $G'$, hence $G$, is solvable. Thus, $G'$ is a locally finite $p$-group, since it is a solvable $p$-group. Also, as in [2], if it is proved that $G$ is an FC-group, then $G$ is $p$-Abelian by the above argument. Perhaps our results lead to a different motivation.

REFERENCES


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