The analysis of a finite-buffer general input queue with batch arrival and exponential multiple vacations

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Abstract: Vacation queueing models have wide range of application in several areas including computer-communication, and manufacturing systems. A finite-buffer single-server queue with renewal input and multiple exponential vacations has been analysed by Karaesmen and Gupta (1996). In this paper we extend the analysis to cover the batch arrivals, i.e. we consider a batch arrival single-server queue with renewal input and multiple exponential vacations. Using the imbedded Markov chain and supplementary variable techniques we obtain steady-state distribution of number of customers in the system at pre-arrival and arbitrary epochs. The Laplace-Stieltjes transforms of the actual waiting-time distribution of the first-, arbitrary- and last-customer of a batch under First-Come-First-Serve discipline have been derived. Finally, we present useful performance measures of interest such as probability of blocking, average queue (system) length. Some tables and graphs showing the effect of model parameters on key performance measures are presented.

Keywords: batch arrival; finite-buffer; multiple vacations; queue; waiting time.

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1 Introduction

There has been a great interest in analysing batch arrival queues during the last three and a half decades, both from theoretical and practical points of view. These queues are frequently encountered in real applications, details of which can be found in Chaudhry and Templeton (1983). The batch arrival finite-capacity queue with server vacation is now common in several applications including telecommunication system, computer-communication network, etc. For example, a processor (server) has secondary jobs (customers) to be performed aside from the primary jobs. The processor performing secondary job when no primary jobs are available corresponds to server is on vacation from primary job. In queueing terminology such queues are known as vacation queues.

Though there are vast amount of literature available on vacation queues with infinite buffer, see e.g. Doshi (1986) and Takagi (1991), very little seems to have been done on the corresponding finite-buffer queues. In the past, most of the research in this direction was restricted to $M/G/1/N$ or $MAP/G/1/N$ type queues, see for example, Lee (1984), Blondia (1991), Takagi (1993), Frey and Takahashi (1997), Niu and Takahashi (1999) and Gupta and Sikdar (2006). Recently, Frey and Takahashi (1999) and Niu, Shu and
Takahashi (2003) have analysed the $M/G/1/N$ and $BMAP/G/1/N$ vacation queue, respectively. Few researchers have also considered batch service queues with vacations, see e.g. Choi and Han (1994), Gupta and Sikdar (2004), Sikdar and Gupta (2005), etc. But in several practical applications it is seen that the input process of the queueing system may not be Poisson or Markovian arrival process rather it may be a more general arrival process. In view of this some researchers have analysed such queues with renewal input. The $GI/M/1$ queue with multiple vacations have been independently studied by Tian, Zhang and Cao (1989) and Chatterjee and Mukherjee (1990). Recently, Chae and Kim (2006) and Chae, Lee and Lee (2007) have considered the $GI/M/1$ queue with multiple and single vacation. In the former, they obtained the transform of the joint distribution of the length of a busy period, the number of customers served during busy period, and the residual inter-arrival time at the instant the busy period ends. In the latter one, they showed that both the queue length and the waiting-time can be stochastically decomposed into meaningful quantities. However, a very little attention have been paid on the corresponding finite-buffer queue. The first study on this direction was due to Karaesmen and Gupta (1996). They consider $GI/M/1/N$ queue with multiple vacations and obtained the system length distributions and waiting time.

In this paper, we consider a $GI^I/M/1/N$ queue with multiple vacations, the vacation times are being assumed to be exponentially distributed, and obtain the distributions of number of customers in the system at pre-arrival and arbitrary epochs. Analysis of the waiting-time in the queue and some important performance measures, such as, probability of blocking, average queue length, etc. along with numerical results are presented. It may be remarked here that Niu, Shu and Takahashi (2003) presented a detailed analysis of the $BMAP/G/1/N$ queue with single and multiple vacation along with set up and closed down time. But from their analysis, one cannot get the result of $D^I/M/1/N$ queue which is a very important model from application point of view in telecommunication system and other related areas. The results of this model can be easily obtained from the queueing model presented in this paper.

## 2 Description of the model

Consider a finite-buffer $GI^I/M/1$ queue with multiple vacations where customers arrive in batches following a renewal process with rate $\lambda$. The batch size $X$ is a random variable (r.v.) with probability mass function $P(X = i) = b_i$ ($i = 1, 2, 3, \ldots$), $b_0 = 0$ and mean batch size $E(X) = \bar{X}$. The server is allowed to take multiple vacations whenever the system has been emptied. On return from a vacation, if the server finds the system non-empty it will start serving the customers present in the queue. Otherwise, if the server finds the system empty it will again go for a vacation and continue in this manner until it finds at least one customer in the queue.

The inter-arrival times of the batches are independent and identically distributed (i.i.d.) r.v.s, with cumulative Distribution Function (DF) $A(x)$, probability density function $a(x)$ and Laplace-Stieltjes Transform (LST) $A^*(\theta)$. The mean inter-arrival time is $a = A^*(1)$ (say). Service ($S$) and vacation times ($V$) are independent and assumed to be exponentially distributed with rate $\mu$ and $\gamma$ respectively. The system has finite-buffer of size $N$ including the one who is in service. The batches which upon arrival unable to find enough space in the buffer for all the members of the batch are, either fully
rejected, or a part of the batch is rejected. Some queueing protocols are based on the
former strategy and is known as total batch rejection policy. Latter one is known as the
partial batch rejection policy. As the partial batch rejection policy utilizes the buffer space
optimally, we consider only this policy. The traffic intensity is given by \( \rho = \frac{b}{a \mu} \).

The state of the system at time \( t \) is described by the following r.v.s., namely
- \( \xi(t) = [1] \{0 \} \) if the server is [busy] [on vacation],
- \( N(t) = \) number of customers present in the system including the one who is in
  service,
- \( \hat{\lambda}(t) = \) the remaining inter-arrival time of a batch who is yet to arrive.

We define the joint probability densities of the system length \( N_x(t) \), state of the server \( \xi(t) \)
and the remaining inter-arrival time \( \hat{\lambda}(t) \), respectively by
\[
\begin{align*}
\pi_{1,n}(x; t) dx &= P(N_x(t) = n, x < \hat{\lambda}(t) \leq x + dx, \xi(t) = 1), 1 \leq n \leq N, x \geq 0. \\
\pi_{0,n}(x; t) dx &= P(N_x(t) = n, x < \hat{\lambda}(t) \leq x + dx, \xi(t) = 0), 0 \leq n \leq N, x \geq 0.
\end{align*}
\]

As we shall discuss the model in limiting case, i.e. when \( t \to \infty \) the above probabilities
will be denoted by \( \pi_{1,n}(x) \) and \( \pi_{0,n}(x) \), respectively.

3 Analysis of the model

In this section, we shall carry out the analytic analysis of \( G^{\frac{1}{2}}/M^V/N \) queue with multiple
vacations and obtain the system-length distribution at various epochs.

3.1 System-length distribution at pre-arrival epoch

We now consider the state of the system just before an arrival of a batch which are
chosen as embedded points. Let \( t_0, t_1, t_2, \ldots \) be the time epochs at which the arrivals occur
and \( t_n^* \) denote the time epochs just before the arrival instant \( t_n \). The inter-arrival times
\( T_{n+1} = t_{n+1} - t_n \), \( n = 0, 1, 2, \ldots \) are i.i.d.r.v.s. with common distribution function \( A(x) \). The
state of the system at \( t_n^* \) is defined as \( \{N_x(t_n^*), \xi(t_n^*)\} \) where \( N_x(t_n^*) \) is the number
of customers in the system, and \( \xi(t_n^*) \) indicates the server’s state which may be busy or
on vacation. In the limiting case, i.e. when \( t \to \infty \), \( \{N_x(t_n^*), \xi(t_n^*)\} \) forms an embedded
Markov chain with state space \( Q = \{(k, 0); 0 \leq k \leq N\} \cup \{(k, 1); 1 \leq k \leq N\} \). Thus, in
limiting case, we have
\[
\begin{align*}
\pi_{1,n}^{*} &= \lim_{t \to \infty} P(N_x(t_n^*) = n, \xi(t_n^*) = 1), 1 \leq n \leq N, \\
\pi_{0,n}^{*} &= \lim_{t \to \infty} P(N_x(t_n^*) = n, \xi(t_n^*) = 0), 0 \leq n \leq N,
\end{align*}
\]

where \( \pi_{1,n}^{*}(\pi_{0,n}^{*}) \) represents the probability that there are \( n \) customers in the system prior
to an arrival epoch of a batch when the server is in the service period (vacation period).
The analysis of a finite-buffer general input queue

Let \( g_k \) be the probability that exactly the \( k \) customers are served during an inter-arrival time of a batch when service period is going on. Similarly, \( h_k \) represents the probability that the \( k \) customers are served during the end of a vacation and an arrival of the next batch. Hence for all \( k \geq 0 \), we have

\[
g_k = \int_0^\infty \left( \frac{\mu(x-u)^k}{k!} e^{-\mu x} \right) \mathrm{d}A(x), \quad \text{and} \quad h_k = \int_0^\infty \left( \frac{\mu(x-u)^k}{k!} e^{-\mu x} \right) y^k e^{-\mu y} \mathrm{d}A(x).
\]

Let \( G(z) \) and \( H(z) \) be the probability generating functions of \( g_k \) and \( h_k \), respectively, and are given by

\[
G(z) = \sum_{k=0}^\infty g_k z^k = A^*(\mu - \mu z) \quad \text{and} \quad H(z) = \sum_{k=0}^\infty h_k z^k = \frac{\gamma}{\gamma - \mu(1-z)} \left[ A^*(\mu - \mu z) - A^*(\gamma) \right].
\]

Again, let \( \omega \) be the probability that the remaining vacation time exceeds the inter-arrival time. As remaining vacation time has memoryless properly, we henceforth denote it by \( V \). Therefore,

\[
\omega = \int_0^\infty e^{-\omega x} \mathrm{d}A(x) = A^*(\gamma).
\]

Observing the system at two consecutive embedded points, we have the Transition Probability Matrix (TPM) \( P \) with four block matrices of the form

\[
P = \begin{bmatrix}
VV_{(N-1)\times N} & VS_{(N-1)\times N} \\
SV_{N\times (N-1)} & SS_{N\times N}
\end{bmatrix}_{(2N+1)\times (2N+1)}.
\]

The first block \( VV \) gives the probability of transitions from vacation-state to vacation-state. The elements of this block are of the form

\[
VV_{(k,0),(k',0)} = \begin{cases}
\sigma b_{k,k'} & 0 \leq k \leq N-1, 1 \leq l \leq N-1, l > k, \\
\sigma b_{k,k'}^L & 0 \leq k \leq N-1, l = N, \\
\sigma b_{k,k'}^0 & 0 \leq k \leq N, l = 0, \\
0, & \text{otherwise},
\end{cases}
\]

where \( b_k = \sum_{i=0}^N b_i \). One may note here that \( VV_{0,0} \) indicates transition from state \( 0 \) to \( 0 \); similar notations are used for other blocks also. One of the other block \( VS \) of the TPM gives the probability of transitions from vacation-state to service-state. The structure of this block is given by

\[
VS_{(k,0),(l,1)} = \begin{cases}
\sum_{r=0}^{N-k} h_{k+r,l} + h_{N-k,l}, & 0 \leq k \leq N-1, 1 \leq l \leq N, l > k, \\
\sum_{r=1}^{N-k} h_{k+r,l} + h_{N-k,l}, & 0 \leq k \leq N-1, 1 \leq l \leq N, l < k, \\
V_{S(k-1)(l,1)}, & k = N, 1 \leq l \leq N.
\end{cases}
\]
SV of the TPM gives the probability of transitions from service-state to vacation-state. This block matrix is of the form

\[
SV_{(k, j| j, l)} = \begin{cases} 
M(k) & 1 \leq k \leq N, l = 0, \\
0 & \text{otherwise.}
\end{cases}
\]

The last block SS of TPM gives the probability of transitions from service-state to service-state. This block is of the form

\[
SS_{(k, j| i, j)} = \begin{cases} 
\sum_{n=1}^{N-k} b_{N-n} g_{N-n} b_{N-n+1} & 1 \leq k \leq N-1, 1 \leq i \leq N, l > k, \\
\sum_{n=1}^{N-k} b_{N-n} g_{N-n} b_{N-n+1} & 1 \leq k \leq N-1, 1 \leq i \leq N, l = k, \\
SS_{(k-1, l| i, j)} & k = N, 1 \leq l \leq N,
\end{cases}
\]

where

\[
L(k) = 1 - \sum_{i=1}^{N} \left[ SV_{(k, j| j, i)} + VS_{(k, j| j, l)} \right],
\]

\[
M(k) = 1 - \sum_{i=1}^{N} SS_{(k, j| i, j)}.
\]

Now the pre-arrival epoch probabilities \( \pi_{0,n}(0 \leq n \leq N) \) and \( \pi_{0,n}(1 \leq n \leq N) \) can be obtained by solving the system of equations \( \pi^* = \pi^* P \), where \( \pi^* = (\pi_{0,0}, \pi_{0,1}, \pi_{0,2}, \ldots, \pi_{0,N}, \pi_{1,1}, \pi_{1,2}, \pi_{1,3}, \ldots, \pi_{1,N}) \). We have used an algorithm developed by Grassmann, Taksar and Heyman (1985) for solving the system of equations.

### 3.2 System-length distribution at arbitrary epoch

To obtain the system-length distribution at arbitrary epoch we develop relations between distributions of number of customers in the system at pre-arrival and arbitrary epochs. For this, we use the supplementary variable technique and relate the state of the system at two consecutive time epochs \( t \) and \( t + \Delta t \) and using probabilistic arguments, in steady-state, we obtain, the following system of differential-difference equations:

\[
-a_{n,j,1}(x) = -\mu \pi_{1,n}(x) + \mu \pi_{1,n+1}(x) + \gamma \pi_{0,1}(x), \quad (1)
\]

\[
-a_{n,j,2}(x) = -\mu \pi_{1,n}(x) + \mu \pi_{1,n+1}(x) + \alpha(x) \sum_{j=1}^{N-1} \pi_{1,n}(0) b_{N-j} + \gamma \pi_{0,n}(x), \quad 2 \leq n \leq N-1, \quad (2)
\]

\[
-a_{n,j,N}(x) = -\mu \pi_{1,n}(x) + \alpha(x) \pi_{1,n}(0) + \alpha(x) \sum_{j=1}^{N-1} \pi_{1,n}(0) \sum_{j=N-n}^{N-1} b_{j} + \gamma \pi_{0,n}(x), \quad (3)
\]

\[
-a_{n,j,0}(x) = \mu \pi_{1,n}(x), \quad (4)
\]
The analysis of a finite-buffer general input queue

\[-\pi_{0,n}^{(1)}(x) = -\gamma \pi_{0,n}(x) + a(x) \sum_{i=0}^{n-1} \pi_{0,i}(0) b_{n-i}, \quad 1 \leq n \leq N - 1.\]  

\[-\pi_{0,N}(x) = -\gamma \pi_{0,N}(x) + a(x) \pi_{0,N}(0) + a(x) \sum_{i=0}^{N-1} \pi_{0,i}(0) \sum_{j=N-n}^{x} b_j, \quad \text{where } \pi_{0,N}(0) \text{ and } \pi_{0,n}(0) \text{ are respective probabilities with remaining inter-arrival time equal to zero, i.e., an arrival is about to occur, and } \pi_{0,n}(x) = \frac{d}{dx} \pi_{1,n}(x).\]

Let us define the Laplace transforms of \(\pi_{1,n}(x), 1 \leq n \leq N, \) and \(\pi_{0,n}(x), 0 \leq n \leq N\) as

\[\pi_{1,n}^*(\theta) = \int_{0}^{\infty} e^{-\theta x} \pi_{1,n}(x) dx \quad \text{and} \quad \pi_{0,n}^*(\theta) = \int_{0}^{\infty} e^{-\theta x} \pi_{0,n}(x) dx.\]

It follows from above that

\[\pi_{1,n}^*(0) = \pi_{1,n} = \int_{0}^{\infty} \pi_{1,n}(x) dx \quad \text{and} \quad \pi_{0,n}^*(0) = \pi_{0,n} = \int_{0}^{\infty} \pi_{0,n}(x) dx,\]

where \(\pi_{1,n}\) is the joint probability that there are \(n \ (1 \leq n \leq N)\) customers in the system while service period is going on. Similarly, \(\pi_{0,n}\) is the joint probability that there are \(n \ (0 \leq n \leq N)\) customers in the system and the server is on vacation.

Multiplying Equations (1–6) by \(e^{\theta x}\) and integrating with respect to \(x\) over 0 to \(\infty\) after using the definition of Laplace transform we get

\[(\mu - \theta) \pi_{1,1}^*(\theta) = \mu \pi_{1,1}^*(\theta) + \gamma \pi_{0,1}^*(\theta) - \pi_{1,1}^*(0).\]  

\[(\mu - \theta) \pi_{1,n}^*(\theta) = \mu \pi_{1,n}^*(\theta) + A^*(\theta) \sum_{i=0}^{n-1} \pi_{1,i}(0) b_{n-i} - \gamma \pi_{0,n}^*(\theta) - \pi_{1,n}^*(0), \quad 2 \leq n \leq N - 1.\]

\[(\mu - \theta) \pi_{1,N}^*(\theta) = A^*(\theta) \pi_{1,N}(0) + A^*(\theta) \sum_{i=0}^{N-1} \pi_{1,i}(0) \sum_{j=N-n}^{x} b_j + \gamma \pi_{0,N}^*(\theta) - \pi_{1,N}^*(0).\]

\[-\theta \pi_{0,1}^*(\theta) = \mu \pi_{1,1}^*(\theta) - \pi_{0,0}^*(0).\]

\[(\gamma - \theta) \pi_{0,n}^*(\theta) = A^*(\theta) \sum_{i=0}^{n-1} \pi_{0,i}(0) b_{n-i} - \pi_{0,n}(0), \quad 1 \leq n \leq N - 1,\]

\[(\gamma - \theta) \pi_{0,N}^*(\theta) = A^*(\theta) \pi_{0,N}(0) - A^*(\theta) \sum_{i=0}^{N-1} \pi_{0,i}(0) \sum_{j=N-n}^{x} b_j - \pi_{0,N}(0).\]

Now using the above equations we derive certain results which will be used later in developing relations among system-length distributions at various epochs.

**Theorem 3.1** The mean number of entrances into the system per unit time equals the mean arrival rate, i.e.,
\[ \sum_{i=1}^{N} \pi_{i,n}(0) + \sum_{i=0}^{N} \pi_{0,i}(0) = \frac{1}{a}. \]

**Proof:** Adding Equations (7-12), we get
\[ \sum_{i=1}^{N} \pi_{i,n}(\theta) + \sum_{i=0}^{N} \pi_{0,i}(\theta) = \frac{1 - A^{*}(\theta)}{\theta} \left[ \sum_{i=1}^{N} \pi_{i,n}(0) + \sum_{i=0}^{N} \pi_{0,i}(0) \right]. \]

Taking the limit as \( \theta \to 0 \) in the above and using the normalising condition:
\[ \sum_{i=1}^{N} \pi_{i,0} + \sum_{i=0}^{N} \pi_{0,i} = 1, \]
after simplification we get the desired result.

### 3.2.1 Relation between system-length distribution at an arbitrary and a pre-arrival epochs

In order to obtained the relation between system-length distribution at an arbitrary and a pre-arrival epochs, we first connect pre-arrival epoch probabiliites \( \pi_{i,n}(\pi_{0,n}) \) and \( \pi_{i,n}(0)(\pi_{0,n}(0)) \). This is given by
\[
\pi_{i,n} = \frac{\pi_{i,n}(0)}{\sum_{i=1}^{N} \pi_{i,n}(0) + \sum_{i=0}^{N} \pi_{0,i}(0)} = a \pi_{i,n}(0), \quad 1 \leq n \leq N, \tag{13}
\]
\[
\pi_{0,n} = \frac{\pi_{0,n}(0)}{\sum_{i=1}^{N} \pi_{i,n}(0) + \sum_{i=0}^{N} \pi_{0,i}(0)} = a \pi_{0,n}(0), \quad 0 \leq n \leq N. \tag{14}
\]

The above result can be easily obtained by the application of Bayes' theorem. As stated earlier, our objective is to obtain the distributions of the number of customers in the system at arbitrary epoch when the server is busy or on vacation, \( \pi_{i,n} (1 \leq n \leq N) \) and \( \pi_{0,n} (0 \leq n \leq N) \). We obtain them the following theorem.

**Theorem 3.2** The relation between pre-arrival \( \pi_{i,n}(\pi_{0,n}) \) arbitrary \( \pi_{i,n}(\pi_{0,n}) \) epoch probabilities is given by
\[
\pi_{0,n} = \frac{1}{\alpha} \left( \sum_{i=0}^{N} \pi_{0,i} - \pi_{0,n} \right), \quad 1 \leq n \leq N - 1, \tag{15}
\]
\[
\pi_{0,N} = \frac{1}{\alpha} \sum_{i=0}^{N-1} \pi_{0,i} \sum_{i=N-n}^{\infty} b_i, \tag{16}
\]
\[
\pi_{1,n} = \frac{\rho}{b} \sum_{i=1}^{N-1} \pi_{1,i} \sum_{i=N-n}^{\infty} b_i + \frac{1}{\mu} \pi_{0,N}, \tag{17}
\]
\[
\pi_{1,n} = \pi_{1,n-1} + \frac{\rho}{b} \left( \sum_{i=1}^{N-n+1} \pi_{1,i} b_{i-1} \right) + \frac{1}{\mu} \pi_{0,n}, \quad n = N-1, N-2, N-3, \ldots, 3, \tag{18}
\]
The analysis of a finite-buffer general input queue

\[ \pi_{i,1} = \pi_{i,2} - \frac{\rho}{b} \pi_{i,1} - \frac{\gamma}{\mu} \pi_{0,0}. \]  

Finally, the only unknown quantity \( \pi_{0,0} \) is obtained by using the normalising condition, i.e.,

\[ \pi_{0,0} = 1 - \sum_{i=1}^{N} (\pi_{i,1} + \pi_{i,0}). \]

**Proof:** Setting \( \theta = 0 \) in Equations (11–12) and Equations (7–9), we get

\[ \pi_{0,n} = \frac{1}{\gamma} \sum_{i=0}^{n-1} \pi_{0,i}(0) b_{n-i} - \frac{1}{\gamma} \pi_{0,n}(0), \quad 1 \leq n \leq N - 1. \]  

(20)

\[ \pi_{N,N} = \frac{1}{\gamma} \sum_{i=0}^{N-1} \pi_{N,i}(0) \sum_{j=i+1}^{N} b_j. \]  

(21)

\[ \pi_{1,1} = \pi_{1,2} + \frac{\gamma}{\mu} \pi_{0,1} - \frac{1}{\mu} \pi_{1,0}(0). \]  

(22)

\[ \pi_{1,n} = \pi_{1,n-1} + \frac{1}{\mu} \left( \sum_{i=1}^{n-1} \pi_{1,i}(0) b_{n-i} - \pi_{1,n}(0) \right) + \frac{\gamma}{\mu} \pi_{0,n}, \quad 2 \leq n \leq N - 1. \]  

(23)

\[ \pi_{1,N} = \frac{1}{\mu} \sum_{i=1}^{N-1} \pi_{1,i}(0) \sum_{j=N-i}^{N} b_j + \frac{\gamma}{\mu} \pi_{0,N}. \]  

(24)

After some manipulation, using Equations (13–14) and \( \rho = \frac{b}{a \mu} \), we obtain the result.

4 Performance measure

Once the distribution of number of customers in the system at various epochs are known, the performance measures such as the average queue length

\[ L_q = \sum_{i=0}^{N} (i-1) \pi_{i,1} + \sum_{i=1}^{N} \pi_{i,0}, \]

average system length \( L = \sum_{i=1}^{N} (\pi_{i,0} + \pi_{0,i}) \) and the blocking probability of the first \((P_{BF})\), an arbitrary \((P_{BA})\) and the last \((P_{BL})\) customer of an arriving batch can be obtained and they are given by

(i) Blocking probability of the first customer

The blocking probability of the first customer of an arriving batch is given by

\[ P_{BF} = \pi_{1,N} - \pi_{0,N}. \]

(ii) Blocking probability of an arbitrary customer of a batch

Let the r.v. \( B \) denote the number of customers before an arbitrary customer within a batch. The distribution of \( B \) is given by, see Takagi (1993)
Hence, the blocking probability of an arbitrary customer of a batch \( P_{BA} \) is given as

\[
P_{BA} = \sum_{r=1}^{\infty} \pi_{1,r} \sum_{j=N-r}^{\infty} b_j + \sum_{r=0}^{\infty} \pi_{0,r} \sum_{j=0}^{\infty} b_j.
\]

(iii) Blocking probability of the last customer of a batch

The blocking probability of the last customer of a batch is given by

\[
P_{BL} = \sum_{r=0}^{N} \pi_{1,r} \sum_{j=N-r}^{\infty} b_j + \sum_{r=0}^{\infty} \pi_{0,r} \sum_{j=0}^{\infty} b_j.
\]

5 Waiting-time analysis

In this section, we carry out the waiting-time analysis of the first-, an arbitrary- and the last-customer of an arriving batch which is accepted under the FCFS service discipline. Through our analysis, we also obtain the average waiting-time in the queue in all the three cases. Let \( W_{q_{1}}(\theta) \) be the LST of the distribution function of the waiting-time in the queue of the first customer of a batch who is accepted in the system. Due to the memoryless property of the service and vacation time distributions, an arrival may find the system in any one of the following two cases:

Case 1: The test customer of a batch who arrives while the server is on vacation and finds \( n \) (\( 0 \leq n \leq N-1 \)) customers in the system, has to wait in the queue till the server returns from a vacation to start service and completes service of \( n \) customers.

Case 2: The test customer of a batch who arrives while the server is busy and finds \( n \) (\( 0 \leq n \leq N-1 \)) customers in the system, has to wait in the queue till the server completes service of \( n \) customers. Combining the above two cases, we have

\[
W_{q_{1}}(\theta) = \frac{1}{1-P_{BA}} \left[ \sum_{r=0}^{N} \pi_{1,r} (B^*(\theta))^{r+1} \left( \frac{\gamma}{\gamma + \theta} \right) + \sum_{r=0}^{\infty} \pi_{0,r} (B^*(\theta))^{r+1} \right],
\]

where \( B^*(\theta) = \mu/(\mu + \theta) \) is the LST of the service time distribution.

Similarly, let \( W_{q_{a}}(\theta) \) and \( W_{q_{l}}(\theta) \) be the LST of the distribution function of the waiting-time in the queue of an arbitrary and the last customer of a batch who is accepted in the system. Proceeding as above, we can easily derive the expressions for \( W_{q_{a}}(\theta) \) and \( W_{q_{l}}(\theta) \) and are given by

\[
W_{q_{a}}(\theta) = \frac{1}{1-P_{BA}} \left[ \sum_{r=0}^{N} \pi_{a,r} \sum_{j=0}^{N-r-1} b_j (B^*(\theta))^{r+1} \left( \frac{\gamma}{\gamma + \theta} \right) - \sum_{r=0}^{\infty} \pi_{0,r} \sum_{j=0}^{\infty} b_j (B^*(\theta))^{r+1} \right].
\]
The analysis of a finite-buffer general input queue

\[ W_q^*(\theta) = \frac{1}{1 - \rho_{\text{M/D}}} \left[ \sum_{r=0}^{N-1} \sum_{j=1}^{N-r} \frac{b_j(\beta^*(\theta))^{r-1}(\frac{\gamma}{\gamma + \beta})}{r!} + \sum_{r=1}^{N} \sum_{j=1}^{N-r} \sum_{j=1}^{N-r} b_j(\beta^*(\theta))^{r-1} \right], \]

where \( b^*_j \) is defined in Section 4. Now the average waiting times in the queue of the first- \((W_{q1})\), an arbitrary- \((W_{qA})\), and the last-customer \((W_{qL})\) can be easily obtained and are given by

\[ W_{q1} = \frac{1}{1 - \rho_{\text{M/D}}} \left[ \sum_{r=0}^{N-1} \sum_{j=0}^{N-r} \frac{\pi_{0,r} b_j(r+1) \left( \frac{r+1}{\mu} \right)}{\mu} + \frac{1}{\mu} \sum_{r=1}^{N-1} \sum_{j=1}^{N-r} b_j(r+j) \right]. \]  

\[ W_{qA} = \frac{1}{1 - \rho_{\text{M/D}}} \left[ \sum_{r=0}^{N-1} \sum_{j=0}^{N-r} \frac{\pi_{0,r} b_j(r+1) \left( \frac{r+1}{\mu} \right)}{\mu} + \frac{1}{\mu} \sum_{r=1}^{N-1} \sum_{j=1}^{N-r} b_j(r+j) \right]. \]  

\[ W_{qL} = \frac{1}{1 - \rho_{\text{M/D}}} \left[ \sum_{r=0}^{N-1} \sum_{j=0}^{N-r} \frac{\pi_{0,r} b_j(r+j-1) \left( \frac{r+j-1}{\mu} \right)}{\mu} + \frac{1}{\mu} \sum_{r=1}^{N-1} \sum_{j=1}^{N-r} b_j(r+j-1) \right], \]

where we have used the fact that \(-\beta^*(\theta)(0) = 1/\mu\) is the mean service time.

The average waiting-time in the system \((W)\) and in the queue \((W_q)\) of an arbitrary customer of a batch can also be obtained using Little's rule and is given as

\[ W = \frac{L}{\lambda'} \quad \text{and} \quad W_q = \frac{L_q}{\lambda'}, \]

where \( \lambda' = b(1 - \rho_{\text{M/D}})/a \), is the effective arrival rate.

It may be noted here that the mean waiting-time in the queue of an arbitrary customer of a batch obtained from Equation (26) is the same as the one obtained using the Little's rule as it should be. This has been checked numerically.

6 Numerical results

We have performed extensive numerical work by considering various inter-arrival time distributions viz: exponential \((M)\), Erlang \((E_k)\) with mean of each phase \(1/\lambda_k\) and the mean inter-arrival time \(a = 1/\lambda\), deterministic \((D)\) and hyperexponential \((HE_2)\), with parameters \(\sigma_1, \sigma_2, \lambda_1, \lambda_2\). No difficulty was encountered for any values of model parameters. All calculations have been done in double precision but they are reported here in six decimal places. The notations used in the tables are the same as those defined throughout the paper.

The distribution of the number of customers in the system at pre-arrival and an arbitrary epochs are given in Tables 1–4 for the \(M^k/M/1/N/\text{M/M/V}15/\text{MV}^n\) \((\text{MV}^n\) stands for multiple vacations), \(E_k^q/M/1/12/\text{M/V}^n, D^q/M/1/15/\text{MV}^n, \) and \(HE_2^q/M/10/\text{MV}^n\) queue, respectively. It can be seen from Table 1 that the pre-arrival and an arbitrary epochs probabilities in \(M^k/M/1/15/\text{MV}^n\) queue are equal as it should be. Further, it is observed that \(W_{q1}\) and \(W_{qA}\) obtained using Little's rule are equal. Finally, taking vacation parameter sufficiently large so that mean vacation time tends to zero, we matched our numerical result to non-vacation \(GP^k/M/1/N\) queue, Table 2.2 of Vijaya Laxmi (2001). This is shown in Table 5.
Table 1  Distributions of number of customers in the system at pre-arrival and arbitrary epochs for $M/\mu, 1/15/\mu$ with $\mu = 1.2, \gamma = 0.5, a = 3.333333, b_1 = 0.3, b_2 = 0.1, b_3 = 0.2, b_4 = 0.2, \bar{b} = 3.6$, and $\rho = 0.9$

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Table 2  Distributions of number of customers in the system at pre-arrival and arbitrary epochs for $E/\mu^0/12/\mu^0$ with $\mu = 1.76, \gamma = 0.2, a = 2.5, b_1 = 0.4, b_2 = 0.2, b_3 = 0.2, b_4 = 0.2, \bar{b} = 2.2$, and $\rho = 0.5$

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The analysis of a finite-buffer general input queue

Table 3  Distributions of number of customers in the system at pre-arrival and arbitrary epochs for $D^s_1/M/1:15/M^V$ with $\mu = 2.0, \gamma = 0.7, \alpha = 4.062563, b_1 = 0.2, b_2 = 0.3, b_3 = 0.5, b_4 = 3.3,$ and $\rho = 0.406148$

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Table 4  Distributions of number of customers in the system at pre-arrival and arbitrary epochs for $HE^3_1/M/1:10/M^V$ with $\mu = 1.0, \gamma = 0.1, \alpha = 2.666670 (\sigma_1 = 0.2, \sigma_2 = 0.8, \sigma_3 = 0.5, \sigma_4 = 4.1,$ and $\rho = 1.537498$

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Table 5  Distributions of number of customers in the system at pre-arrival and arbitrary epochs for H/E$_2$/M/1/10/MD with $\mu = 1.0$, $\gamma = 1000000$, $\alpha = 2.6667$ ($\sigma_1 = 0.2$, $\sigma_2 = 0.8$).

$\lambda_1 = 0.5$, $\lambda_2 = 0.4$, $b_1 = 0.2$, $b_3 = 0.3$, $b_4 = 0.5$, $\delta = 4.1$, and $\rho = 1.5375$

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<td>0.1514</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.0000</td>
<td>0.1794</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sum</td>
<td>0.0802</td>
<td>0.9198</td>
<td>0.0826</td>
<td>0.917393</td>
</tr>
</tbody>
</table>

$L_\infty = 5.4258$, $L = 6.3432$, $P_{BH} = 0.1794$, $P_{BA} = 0.4033$, $P_{BE} = 0.5613$, $W_{\infty} = 5.5810$, $W_{\infty} = 5.9143$, $W_{\infty} = 6.3109$, $W = 6.9143$, $W = 5.9143$.

Figure 1  (a) $N$ vs. average waiting time; (b) $N$ vs. probability of blocking

Figure 2  (a) $\rho$ vs. $P_{BA}$; (b) $\rho$ vs. $L$
In Figure 1(a,b), we have plotted average waiting-time ($W_{ij}$, $W_{ij}$, $W_{ij}$) and probability of blocking ($P_{BA}$, $P_{BF}$, $P_{BE}$) against buffer size ($N$), respectively, in $E_{ij}^p / M / 1 / N / MV$ queue where $N$ varies from 5 to 50 with $\lambda = 0.4$, $\gamma = 0.2$, $\mu = 1.76$, $b_1 = 0.4$, $b_2 = 0.2$, $b_3 = 0.2$, $b_4 = 0.2$ and $\rho = 0.5$. From Figure 1(a), it can be seen that as $N$ increases average waiting-time increases initially, but finally it becomes constant with further increases of $N$. This is due to the fact that the model behaves as an infinite buffer queue. It can be observed from Figure 1(b) that as $N$ increases, probabilities of blocking $P_{BA}$, $P_{BF}$ and $P_{BE}$ asymptotically approaches to zero.

In Figure 2(a,b), we have plotted probability of blocking $P_{BA}$ and average system length against traffic intensity, respectively, in a $GI^{p} / M / 1 / 5 / MV$ queue with $\lambda = 0.272728$, $\gamma = 0.5$, $b_1 = 0.2$, $b_2 = 0.3$ and $b_3 = 0.5$. Four types of inter-arrival time distributions ($D$, $M$, $HE_2$ and $E_1$) are assumed. From Figure 2(a) it is seen that up to certain level of increase in traffic intensity (say, $\rho = 0.6$) $P_{BA}$ is quite low for $D$, $E_1$ and $HE_2$ distributions, and after that $P_{BA}$ takes a drastic linear increase as $\rho$ increases. But for exponential distribution ($M$), initially $P_{BA}$ decreases and after that it increases as $\rho$ increases. From Figure 2(b) it can be observed that $L$ almost linearly increases as $\rho$ increases.

7 Conclusions

In this contribution, we have discussed analytical and computational aspects of batch arrival single-server queue with renewal input and multiple vacations. The steady-state distribution of the number of customers in the system at pre-arrival and arbitrary epochs have been obtained. Finally, we have derived a few important system characteristics, which includes the blocking probabilities, the average waiting times of the first-, an arbitrary-, and the last-customer of a batch. It may be remarked here that the method of analysis adopted in this paper can be used to analyse other complex models such as the finite-buffer bulk arrival bulk service queue with renewal input and exponential multiple vacations, i.e. $GI^{p} / M^{\mu \rho} / N$.

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References


