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Subventionné par les départements de mathématiques des Universités Queen's, de Toronto et de Waterloo.
ON SOLVABLE LIE IDEALS OF A RING

R.K. Sharma and J.B. Srivastava

Abstract: Let \( R \) be an associative, unitary ring in which \( 2 \) is invertible. It is proved that if a Lie ideal \( U \) of \( R \) is solvable then \( \gamma_2(U)R \) is a two sided nil ideal of \( R \).

Let \( R \) be an associative ring with identity and \( \mathcal{L}(R) \) be the associated Lie ring of \( R \) under the Lie multiplication
\[
[x, y] = xy - yx; \quad x, y \in R.
\]

An ideal \( U \) of \( \mathcal{L}(R) \) is called a Lie ideal of \( R \). The identity
\[
ur = [u, r] + ru; \quad u \in U, r \in R
\]
implies that \( UR = RU = RUR \) is the two sided ideal of \( R \) generated by \( U \). For any two Lie ideals \( U \) and \( V \) of \( R \), \( [U, V] \) denotes the Lie ideal of \( R \) generated by all \( [u, v] \); \( u \in U, v \in V \).

The Commutators are defined left normed, i.e.
\[
[x_1, x_2, \ldots, x_n] = [(x_1, x_2, \ldots, x_{n-1}), x_n], \text{ for } n \geq 3 \quad \text{and} \quad [x_1, x_2] = x_1 x_2 - x_2 x_1
\]
for all \( x_1, x_2, \ldots, x_n \in R \). The derived chain and the lower central chain of a Lie ideal \( U \) of \( R \) are defined by
\[
\delta^{(0)}(U) = U, \quad \delta^{(m)}(U) = [\delta^{(m-1)}(U), \delta^{(m-1)}(U)] \quad \text{for } m \geq 1,
\]
and
\[
\gamma_1(U) = U, \quad \gamma_n(U) = \gamma_{n-1}(U), U \quad \text{for } n \geq 2, \text{ respectively.}
\]

\( U \) is said to be solvable (nilpotent) if for some positive integer \( C \), \( \delta^{(C)}(U) = 0 \) (\( \gamma_{C+1}(U) = 0 \)). \( R \) is said to be Lie solvable (Lie nilpotent) if there exists a positive integer \( n \) such that \( \delta^{(n)}(\mathcal{L}(R)) = 0 \), \( \gamma_{n+1}(\mathcal{L}(R)) = 0 \).
Jennings [1] proved that if a ring \( R \) is Lie nilpotent then \( \gamma_2(\mathcal{Z}(R))R \) is a nil ideal of \( R \). Sharma and Srivastava [3] proved that if a ring \( R \) in which both 2 and 3 are invertible is Lie solvable, then \( \gamma_2(\mathcal{Z}(R))R \) is a nil ideal of \( R \). In case of Lie nilpotent grouprings, we refer to Levin and Sehgal [2] and Sharma and Srivastava [4]. In this paper we take up the case of a solvable Lie ideal \( U \) of a ring \( R \) in which 2 is invertible and prove (Theorem 5) that \( \gamma_2(U)R \) is a two sided nil ideal of \( R \). It is shown that the condition of invertibility of 2 cannot be dropped. Some other related results are also obtained.

We begin with

**Lemma 1**. For any Lie ideal \( U \) of a ring \( R \),

\[
\gamma_3(U)R^3 \leq \delta^3(U)R
\]

**Proof.** follows from Lemma 2.4(11) and Theorem 2.7 of [4].

**Lemma 2**. For any Lie ideal \( U \) of a ring \( R \),

\[
\delta^{(1)}(U), \mathcal{Z}(R) \text{ and } \delta^{(1)}(1) \leq \delta^{(2)}(U)R.
\]

**Proof.** follows from Lemma 1 and Corollary (1.6(1), [3]).

**Lemma 3**. Let \( U \) be a Lie ideal of a ring \( R \), then for \( x, y \in U \), \( 4(x, y)^3 \in \gamma_3(U)R \).

**Proof.** We observe that

\[
2 (x, y)^2 = [x^2, y, y] + x[y, x, y] + [y, x, y]x
\]

\[
\equiv [x^2, y, y] \pmod{\gamma_3(U)R}.
\]

And,

\[
2 (x^2, y, x) = [x^2, y, x] + [(x^2, y, y), [x, y]]
\]

\[
+ [x^2, y, y]y + y[x^2, y, x]
\]

\[
\equiv [y^2, x^2, y, x] \pmod{\gamma_3(U)R}.
\]
But \( [y^2, x^2, y, x] \in [\langle U^2, \mathcal{X}(R) \rangle, U, U] \subseteq [\langle U, \mathcal{X}(R) \rangle, U, U] \)
\( \subseteq \gamma_3(U) \) by Lemma (1.2(1) (3)).

Hence, \( 4(x, y)^3 \in \gamma_3(U)R \)

**Lemma 4.** Let \( U \) be a Lie ideal of a ring \( R \) in which \( 2 \) is invertible.

Then for every \( \alpha \in \delta^2(U)R \) there exists a positive integer \( M \) such that \( \alpha^M \in \delta^2(U)R \).

**Proof.** Let \( x_i, y_i \in U \) and \( r_i \in R \) for \( i = 1, 2, \ldots, n \).

If \( \alpha = \sum_{i=1}^{n} [x_i, y_i]r_i \in \delta^2(U)R \), then \( \alpha^{2n+1} \) will be a finite sum consisting of \((2n+1)\)-fold products of the elements of the type \( \{x_i, y_i\}r_i \), \( i = 1, 2, \ldots, n \), and in each such \((2n+1)\)-fold product at least one \( \{x_j, y_j\}r_j \) for some \( j = 1, 2, \ldots, n \) will be repeated at least 3-times. Hence \( \alpha^{2n+1} \) is a finite sum of the elements of the type

\[ r(x_j, y_j)s(x_j, y_j)t(x_j, y_j)w \]

for some \( r, s, t, w \in R \).

The proof of the lemma follows from the following observation and

**Lemma 2**

\[ r(x_j, y_j)s(x_j, y_j)t(x_j, y_j)w \]
\[ = r(x_j, y_j)s(x_j, y_j)^3tw - r(x_j, y_j)s(x_j, y_j)x_j, y_j, tw \]
\[ = r(x_j, y_j)s(x_j, y_j)^3tw \pmod{\delta^3(U)R(R)R} \]
\[ = rs(x_j, y_j)^3tw + r(x_j, y_j)s(x_j, y_j)^2tw \]
\[ = rs(x_j, y_j)^3tw \pmod{\delta^4(U)R(R)R} \]
\[ = 0 \pmod{\delta^4(U)R(R)R} \] by Lemma 3

\( M \) can be taken as any positive integer greater or equal to \( \delta(2n+1) \).

We can now easily conclude

**Theorem 5.** Let \( R \) be a ring in which \( 2 \) is invertible. If a Lie ideal \( U \) of \( R \) is solvable, then \( \gamma_2(U)R \) is a two sided nil ideal of \( R \).
Proof. follows by repeated applications of Lemma 4.

We can improve upon the Theorem 2.4 of [3] as

Theorem 6. Let $R$ be a ring in which 2 is invertible. If $R$ is Lie solvable then $\gamma_2(2(R))R$ is a two sided nil ideal of $R$.

Proof. follows from Theorem 5, for $U=2(R)$.

Theorem 7. Let $R$ be a ring in which 2 is invertible. If a Lie ideal $U$ of $R$ is nilpotent, then $\gamma_2(U)R$ is a two sided nil ideal of $R$.

Proof. follows from Theorem 5.

Remark 8. The condition of invertibility of 2 in Theorems 5, 6 and 7 can not be dropped, for example, if $R=\mathbb{Z}_2[S_3]$, the group algebra of characteristic 2 of the group of permutations $S_3$ on three symbols over $\mathbb{Z}_2=\{0,1\}$, and $U=\gamma_2(2(R))$, then it is easy to see that $\delta^{(2)}(U) \subseteq \gamma_2(U)=0$, $(\sigma+\sigma^2) \in \gamma_2(U)$ and $(\sigma+\sigma^2)^k=(\sigma+\sigma^2)^k=0$ for every positive integer $k$, where $\sigma=(1,2,3)$.

References.


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ON UNITS IN \( ZD_8 \)

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The derived and lower central chains have been studied for the group of units of augmentation 1 in the integral group ring of the dihedral group of order 8.

1. Introduction

Let \( G \) be a finite group, \( Z \) the ring of integers and \( ZG \) the integral group ring of \( G \). An invertible element \( \sigma = \sum a \sigma \in ZG \) is called a unit of augmentation 1 in \( ZG \) if \( \Sigma a = 1 \).

Let \( D_8 = \langle a, b : a^4 = 1, b^2 = 1, ab = ba^{-1} \rangle \) be the dihedral group of order 8 and let the group of units of augmentation 1 in the integral group ring \( ZD_8 \) be denoted by \( V = V(ZD_8) \). Dennis [1] obtained the following presentation of \( V \) in terms of generators and relations:

\[
V = \langle a, b, A, B, C : a^4 = b^2 = 1, ab = ba^{-1}, aAa^{-1} = B^{-1}, aBa^{-1} = A^{-1}, \\
aCa^{-1} = C^{-1}, bAb = B, bB = A, bC = AC^{-1}B \rangle \quad (\ast)
\]

Here \( \{A, B, C\} \) generates a free group of rank 3 in \( V \). We shall denote it by \( F_3 \). For any \( x, y \in V \), the commutator of \( x \) and \( y \) is denoted by \( (x, y) := xyx^{-1}y^{-1} \). Clearly \( x \) and \( y \) commute if and only if \( (x, y) = 1 \). This fact is also denoted some times by writing \( x \equiv y \), particularly when \( x \) and \( y \) are generators of a group \( G \). Furthermore, if \( x \) commutes with both \( y \) and \( z \), then this fact is denoted by \( x \equiv \{y, z\} \). For any two subsets \( X, Y \) of \( V \), \( (X, Y) := \{(x, y) : x \in X, y \in Y\} \). However, when \( X, Y \) are subgroups of \( V \) then \( (X, Y) \) will mean the subgroup \( \{(x, y) : x \in X, y \in Y\} \). Further, let \( V \supseteq V^{(i)} \supseteq V^{(i+1)} \supseteq \cdots \supseteq V^{(n)} \supseteq \cdots \) and \( V = \gamma_1(V) \supseteq \gamma_2(V) \supseteq \cdots \supseteq \gamma_n(V) \supseteq \cdots \) respectively, be the derived and lower central chains of \( V \).

In this note, we prove that \( V^{(i)} \) is a free group of infinite rank for every \( i \geq 2 \). Further, we prove that \( \gamma_n(V) \) is a free group of rank \( 129 \).

Let the group \( G \) be presented as

\[
(a_1, \ldots, R_1, \ldots)
\]

and \( H \) be a subgroup of \( G \). Let \( K \) be a right coset representative system for \( G \bmod H \) then \( K \) consists of elements of \( G \) each of which comes from a distinct right coset of \( H \) in

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