N-ADDITIVE MAPPING

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Abstract

Let $R$ be a prime ring and $M$ a trace of an $n$-additive mapping from $R^n$ into $S$, the central closure of $R$, such that $M(s, s, \ldots, s)[M(s, s, \ldots, s), s] + [M(s, s, \ldots, s), s]M(s, s, \ldots, s) = 0 \forall s \in R$. Let $\binom{2n}{n}$ be invertible in $C$, the extended centroid of $R$ and suppose char $R = 0$ or char $R > 2n$ then it is proved that there exist mappings $\lambda_i$ of $R$ into $C$ such that $M(s, s, \ldots, s)^2 = \sum_{i=0}^{2n} \lambda_i(s)s^{2n-i}$. Further, if $R$ is not algebraic of bounded degree $\leq 2n$, then it is proved that for each $i = 1, 2, \ldots, 2n$, $\lambda_i(s)$ is an $i$-additive mapping.

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Let $R$ be a prime ring and $C$ be the extended centroid of $R$. Let $A$ be the additive group of $R$. A mapping $M$ from $A^n$ to $R$ is said to be $n$-additive if $M(s_1, s_2, \ldots, s_n) = M(s_1, s_2, \ldots, s_i + t_i, \ldots, s_n) \forall i = 1, 2, \ldots, n$. $M$ is said to be symmetric if $M(s_1, s_2, \ldots, s_n) = M(s_{\sigma(1)}, s_{\sigma(2)}, \ldots, s_{\sigma(n)}) \forall \sigma \in S_n$, the symmetric group of $n$-symbols. We denote the Lie product $st - ts$ of $s$ and $t$ by $[s, t]$ and Jordan product $st + ts$ of $s$ and $t$ by $s \circ t$. The mapping $T_M : A \rightarrow R$ defined by $T_M(s) = M(s, s, \ldots, s)$ is called the trace of $M$.

In this paper we prove the following two main theorems:

**Theorem 1:**

Let $R$ be a prime ring and $M : R^n \rightarrow S$ an $n$-additive mapping such that $M(s, s, \ldots, s)[M(s, s, \ldots, s), s] + [M(s, s, \ldots, s), s]M(s, s, \ldots, s) = 0 \forall s \in R$. Suppose that $\binom{2n}{n}$ is invertible in $C$ and char $R = 0$ or char $R > 2n$. Then for every $s \in R$, there exist $\lambda_0, \lambda_1, \ldots, \lambda_{2n} \in C$ such that $M(s, s, \ldots, s)^2 = \sum_{i=0}^{2n} \lambda_i s^{2n-i}$.

**Theorem 2:**

Let $R$ be a prime ring and $M : R^n \rightarrow S$ an $n$-additive mapping such that $M(s, s, \ldots, s)[M(s, s, \ldots, s), s] + [M(s, s, \ldots, s), s]M(s, s, \ldots, s) = 0 \forall s \in R$. Further, suppose $\binom{2n}{n}$ is invertible in $C$ and char $R = 0$ or char $R > 2n$ and $R$ is not algebraic of bounded degree $\leq 2n$ over $C$. Then for every $s \in R$, there exists a $\lambda_0 \in C$ and mappings $\lambda_i : R \rightarrow C$, $i = 1, 2 \ldots, 2n$ such that each $\lambda_i$ is the trace of an $i$-additive mapping and $M(s, s, \ldots, s)^2 = \sum_{i=0}^{2n} \lambda_i(s) s^{2n-i}$.

Let $A$ and $B$ be two additive abelian groups. Given a mapping $N : A \rightarrow B$ define a mapping $\tilde{N}$ on a nonempty finite subset of $A \times \mathbb{Z}$ inductively

1. $\tilde{N}((s, k)) = N(s)$ for $(s, k) \in A \times \mathbb{Z}$

2. $\tilde{N}(\Omega) = N(\bigcup_{(s, k) \in \Omega} s) - \sum_{(s, k) \in \Omega} \tilde{N}(\Gamma)$ for $\Omega \subseteq A \times \mathbb{Z}$ with $1 < |\Omega| < \infty$.

We shall need the following two results:

**Lemma:** [2, Lemma 1.2] Let $R$ be a prime ring with extended centroid $C$ and central closure $S$. Let $a_i, b_i, c_j, d_j$ be elements in $S$ such that $\sum_{i=1}^m a_i b_i + \sum_{j=1}^n c_j s d_j = 0 \forall s \in R$. If $a_1, a_2, \ldots, a_m$ are linearly independent over $C$, then each $b_i$ is a linear combination of $d_1, d_2, \ldots, d_n$ over $C$.

**Theorem:** [2, Theorem 2.4] Let $A$ and $B$ be additive abelian groups and $n$ a natural number. Suppose that $n!$ is invertible in $B$, that is, for each $s \in B$ there exists a unique $t \in B$ such that $(n!)t = s$. Then a mapping
N : A → B is the trace of some n-additive mapping iff both the following conditions hold:

1. \( N(ks) = k^nN(s) \) for any \( k \in \mathbb{Z} \) and \( s \in A \)

2. \( \bar{N}(\Omega) = 0 \) for any \( \Omega \subseteq A \times \mathbb{Z} \) with \( |\Omega| = n + 1 \).

Let \( M : \mathbb{R}^n \to S \) be an n-additive mapping such that

\[
M(s, s, \ldots, s)[M(s, s, \ldots, s), s] + [M(s, s, \ldots, s), s]M(s, s, \ldots, s) = 0
\]

\( \forall s \in \mathbb{R} \). For \( s = \sum_{i=1}^{2n+1} s_i \) where \( s_i \in \mathbb{R}, i = 1, 2, \ldots, 2n + 1 \) we have

\[
M\left( \sum_{i=1}^{2n+1} s_i, \sum_{i=1}^{2n+1} s_i, \ldots, \sum_{i=1}^{2n+1} s_i \right)[M\left( \sum_{i=1}^{2n+1} s_i, \sum_{i=1}^{2n+1} s_i, \ldots, \sum_{i=1}^{2n+1} s_i \right), \sum_{i=1}^{2n+1} s_i] + [M\left( \sum_{i=1}^{2n+1} s_i, \sum_{i=1}^{2n+1} s_i, \ldots, \sum_{i=1}^{2n+1} s_i \right), \sum_{i=1}^{2n+1} s_i]M\left( \sum_{i=1}^{2n+1} s_i, \sum_{i=1}^{2n+1} s_i, \ldots, \sum_{i=1}^{2n+1} s_i \right) = 0
\]

Linearizing the above relation we get

\[
\sum_{\sigma \in S_{2n+1}} \{M(s_{\sigma(1)}, s_{\sigma(2)}, \ldots, s_{\sigma(n)})[M(s_{\sigma(n+1)}, \ldots, s_{\sigma(2n+1)})] + [M(s_{\sigma(1)}, s_{\sigma(2)}, \ldots, s_{\sigma(n)})], s_{\sigma(n+1)}, \ldots, s_{\sigma(2n+1)}]\} = 0
\]

Equivalently,

\[
\sum_{\sigma \in S_{2n+1}} [M(s_{\sigma(1)}, s_{\sigma(2)}, \ldots, s_{\sigma(n)}) \circ M(s_{\sigma(n+1)}, \ldots, s_{\sigma(2n+1)})] = 0 \quad (1)
\]

\[
\sum_{j=1}^{2n+1} \left[ \sum_{\{i_1, i_2, \ldots, i_n\} \subset \{1, 2, \ldots, 2n+1\} \text{ having } n \text{ elements and } \{i_{n+1}, i_{n+2}, \ldots, i_{2n}\} = \{1, 2, \ldots, j, \ldots, 2n+1\} \setminus \{i_1, i_2, \ldots, i_n\}} \right]M(s_{i_1}, s_{i_2}, \ldots, s_{i_n}) \circ M(s_{i_{n+1}}, s_{i_{n+2}}, \ldots, s_{i_{2n}}) = 0
\]

Since \( M \) is assumed to be symmetric and \( \circ \) is commutative, we get \( \frac{1}{2} \binom{2n}{n} \) terms in the summation within
the Lie bracket. Let $t_1, t_2, \ldots, t_{2n}$ be any $2n$ elements of $R$. Define

$$B(t_1, t_2, \ldots, t_{2n}) = \frac{1}{2^{2n}} \sum_{i=1}^{\binom{2n}{2}} M(t_{i_1}, t_{i_2}, \ldots, t_{i_{2n}}) \circ M(t_{i_{n+1}}, t_{i_{n+2}}, \ldots, t_{i_{2n}})$$

where $i_j \in \{1, 2, \ldots, 2n\}$ for $j = 1, 2, \ldots, 2n$ and all $i_j$ are distinct. Clearly $B$ is symmetric and $2n$ additive and (1) can be written as

$$\sum_{i=1}^{2n+1} [B(s_1, s_2, \ldots, s_i, \ldots, s_{2n+1}), s_i] = 0 \quad (2)$$

We are now ready to prove our main theorems:

**Proof of Theorem 1.**

In identity (2) put $s_1 = s_2 = \ldots = s_{2n} = s^2, s_{2n+1} = t$ we get

$$2n[B(s^2, s^2, \ldots, s^2, t), s^2] + [B(s^2, s^2, \ldots, s^2), t] = 0. \quad (3)$$

Set $B_i = B(s, s, \ldots, s, s^2, s^2, \ldots, s^2), B_{i,1} = B(s, s, \ldots, s, s^2, s^2, \ldots, s^2, t)$ and suppose $B_{2n,1} = 0$. Then (3) can be written as

$$2n[B_{0,1}, s^2] + [B_0, t] = 0 \quad (4)$$

Let $f$ be a mapping given by $f(r) = sr + rs \forall s \in R$. Since $f[r, s] = [r, s^2] \forall s \in R$. (4) becomes

$$2nf[B_{0,1}, s] + [B_0, t] = 0 \quad (5)$$

In general, we have

$$(-1)^k \binom{2n}{k} (2n - k) f^{k+1}[B_{k,1}, s] + \sum_{i=0}^{k} (-1)^i \binom{2n}{i} f^i[B_i, t] = 0 \quad (6)$$

for $k = 0, 1, \ldots, 2n$. 
We prove (6) by induction on $k$. For $k = 0$ (6) reduces to (5). Now assume that $0 \leq k \leq 2n - 1$ and (6) holds. Put $s_1 = s_2 = \ldots = s_{2n-k-1} = s^2, s_{2n-k} = \ldots = s_{2n} = s, s_{2n+1} = t$ in (2) to get

$$(2n - k - 1) [B_{k+1,1}, s^2] + (k + 1) [B_{k,1}, s] + [B_{k+1, t}] = 0.$$ 

hence, $[B_{k,1}, s] = -\left(\frac{2n-k-1}{k+1}\right) f[B_{k+1,1}, s] - \frac{1}{k+1} [B_{k+1, t}]$. Put this into (6), we get

$$(-1)^{k} \binom{2n}{k} (2n - k) f^{k+1} \left\{ -\left(\frac{2n-k-1}{k+1}\right) f[B_{k+1,1}, s] - \frac{1}{k+1} [B_{k+1, t}] \right\}$$

$$+ \sum_{i=0}^{k} (-1)^{i} \binom{2n}{i} f^{i}[B_{i}, t] = 0.$$ 

That is,

$$(-1)^{k+1} \binom{2n}{k+1} (2n - k - 1) f^{k+2}[B_{k+1,1}, s] + \sum_{i=0}^{k+1} (-1)^{i} \binom{2n}{i} f^{i}[B_{i}, t] = 0.$$ 

By induction hypothesis (6) holds for all values of $k = 0, 1, \ldots, 2n$. In particular for $k = 2n$, we get from (6)

$$\sum_{i=0}^{2n} (-1)^{i} \binom{2n}{i} f^{i}[B_{i}, t] = 0.$$ 

$$\Rightarrow \sum_{i=0}^{2n} \sum_{j=0}^{i} (-1)^{j} \binom{i}{j} s^{j} [B_{j}, t] s^{i-j} = 0.$$ 

as $f^{i}(r) = \sum_{j=0}^{i} \binom{i}{j} s^{j} r s^{i-j} \forall r \in R$. From here we get

$$\sum_{j=0}^{2n} \Lambda_{j}(s) t s^{j} - \sum_{j=0}^{2n} s^{j} t \psi_{j}(s) = 0 \quad \forall s, t \in R,$$ 

(7)
This gives

\[ \sum_{i=0}^{m} \sum_{j=0}^{m} \alpha_{i+1}s^{i-j}[B_{2n}, t]s^j = 0. \]

Hence

\[ \sum_{j=0}^{m} \Lambda'_j(s)t^j - \sum_{j=0}^{m} s^jY'_j(s) = 0 \quad \forall \ t \in R. \]

where \( \Lambda'_j = \sum_{i=j}^{m} \alpha_{i+1}s^{i-j}B_{2n} \) and \( Y'_j(s) = \sum_{i=j}^{m} \alpha_{i+1}B_{2n}s^{i-j} \) for \( j = 0, 1, \ldots, 2n \).

Now \( 1, s, \ldots, s^m \) are linearly independent over \( C \), hence each \( Y'_j(s) \) is a linear combination of \( 1, s, \ldots, s^m \) by Lemma (1). In particular \( B_{2n} = B(s, s, \ldots, s) = \binom{2n}{n}M(s, s, \ldots, s)^2 = \sum_{i=0}^{m} \mu_is^{m-i} = Y'_m(s) \) i.e. \( M(s, s, \ldots, s)^2 = \sum_{i=0}^{m} \lambda_ix^{m-i} \) for some \( \lambda_i = \frac{1}{\binom{n}{i}}\mu_i \) in \( C \). This proves theorem 1.

**Proof of Theorem 2.**

Put \( s_1 = s_2 = \ldots, = s_{2n} = st \) and \( s_{2n+1} = u \) in (2) we get

\[ 2n[B(st, st, \ldots, u), st] + [B(st, st, \ldots, st), u] = 0, s, t, u \in R. \] (8)

Define

\[ B_{i,j} = B(s, s, \ldots, s, t, t, \ldots, t, st, st, \ldots, st), \]

\[ i \]

\[ j \]

\[ 2n-i-j \]
\[ B_{i,j,1} = B(s, s, \ldots, s, t, t, \ldots, t, st, st, \ldots, st, u). \]

Let \( B_{2n,0,1} = B_{0,2n,1} = 0 \). Then (8) can be written as

\[ 2n[B_{0,0,1}, st] + [B_{0,0}, u] = 0 \]  \hspace{1cm} (9)

In general, we have

\[ (-1)^k \binom{2n}{k} (2n - k) \sum_{j=0}^{k} \binom{k}{j} x^{k-j}[B_{j,k-j,1}, st]t^j \]

\[ + \sum_{i=0}^{k} \sum_{j=0}^{k} (-1)^i \binom{2n}{i} \binom{i}{j} s^{i-j}[B_{j,i-j,1}, u]t^j = 0. \]  \hspace{1cm} (10)

for \( k = 0, 1, \ldots, 2n \). We prove this by induction on \( k \). For \( k = 0 \) it becomes (9). So assume this is true for \( 0 \leq k \leq 2n - 1 \). In equation (2), put

\[ s_1 = s_2 = \ldots = s_j = s, s_{j+1} = \ldots = s_{k+1} = t, s_{k+2} = \ldots = s_{2n} = st \text{ and } s_{2n+1} = u. \]

Then we get

\[ j[B_{j-1,k-j+1,1}, s] + (k - j + 1)[B_{j,k-j,1}, t] \]

\[ + (2n - k - 1)[B_{j,k-j+1,1}, st] + [B_{j,k-j+1,1}, u] = 0. \]

The first part of this summation becomes

\[ (-1)^k \binom{2n}{k} (2n - k) \sum_{j=0}^{k} \binom{k}{j} s^{k-j}[B_{j,k-j,1}, st]t^j \]

\[ = (-1)^k \binom{2n}{k} (2n - k) \sum_{j=0}^{k} \binom{k}{j} \left\{ s^{k-j}[B_{j,k-j,1}, s]t^{j+1} + s^{k-j+1}[B_{j,k-j,1}, t]t^j \right\} \]

\[ = (-1)^k \binom{2n}{k} (2n - k) \sum_{j=0}^{k} \binom{k}{j} s^{k-j}[B_{j,k-j,1}, s]t^{j+1} \]

\[ + (-1)^{k+1} \binom{2n}{k} (2n - k) \sum_{j=1}^{k} \binom{k}{j} \frac{j}{k-j+1} s^{k-j+1}[B_{j-1,k-j+1,1}, s]t^j \]
+ (-1)^{k+1} \binom{2n}{k} (2n-k) \sum_{j=0}^{k} \binom{k}{j} \frac{2n-k-1}{k-j+1} s^{k-j+1}[B_{j,k-j+1,1}, st]t^j
\]

+ (-1)^{k+1} \binom{2n}{k} (2n-k) \sum_{j=0}^{k} \binom{k}{j} \frac{1}{k-j+1} s^{k-j+1}[B_{j,k-j+1,1}, u]t^j

Put \( s_1 = s_2 = \ldots = s_{k+1} = s, s_{k+2} = s_{k+3} = \ldots = s_{2n} = st, s_{2n+1} = u \) in (2). We get \((2n-k-1)[B_{k+1,0,1}, st] + (k+1)[B_{k,0,1}, s] + [B_{k+1,0,1}, st] = 0 \) and then above equation becomes

\[
= (-1)^n \binom{2n}{k} (2n-k) \sum_{j=0}^{k+1} \binom{k}{j} s^{k-j+1}[B_{j-1,k-j+1,1}, s]t^j
\]

+ (-1)^{k+1} \binom{2n}{k} (2n-k) \sum_{j=1}^{k} \binom{k}{j-1} s^{k-j+1}[B_{j-1,k-j+1,1}, s]t^j

+ (-1)^{k+1} \binom{2n}{k+1} (2n-k-1) \sum_{j=0}^{k} \binom{k+1}{j} s^{k-j+1}[B_{j,k-j+1,1}, st]t^j

+ (-1)^{k+1} \binom{2n}{k+1} \sum_{j=0}^{k} \binom{k+1}{j} s^{k-j+1}[B_{j,k-j+1,1}, u]t^j

= (-1)^k \binom{2n}{k} (2n-k)[B_{k,0,1}, s]t^{k+1}

+ (-1)^{k+1} \binom{2n}{k+1} \sum_{j=0}^{k} \binom{k+1}{j} \frac{1}{k-j+1} s^{k-j+1}[B_{j,k-j+1,1}, st]t^j

+ (-1)^{k+1} \binom{2n}{k+1} \sum_{j=0}^{k} \binom{k+1}{j} s^{k-j+1}[B_{j,k-j+1,1}, u]t^j

= (-1)^{k+1} \binom{2n}{k} (2n-k) \frac{2n-k-1}{k+1} [B_{k+1,0,1}, st]t^{k+1}

+ (-1)^{k+1} \binom{2n}{k} (2n-k) \frac{1}{k+1} [B_{k+1,0,1}, u]t^{k+1}
\begin{align*}
&+ (-1)^{k+1} \binom{2n}{k+1} (2n - k - 1) \sum_{j=0}^{k} \left( \binom{k+1}{j} s^{k-j+1} [B_{j,k-j+1,1}, st] t^j \right) \\
&+ (-1)^{k+1} \binom{2n}{k+1} \sum_{j=0}^{k+1} \left( \binom{k+1}{j} s^{k-j+1} [B_{j,k-j+1,1}, u] t^j \right) \\
&= (-1)^{k+1} \binom{2n}{k+1} (2n - k - 1) [B_{k+1,0,1}, st] t^{k+1} + (-1)^{k+1} \binom{2n}{k+1} [B_{k+1,0,1}, u] t^{k+1} \\
&+ (-1)^{k+1} \binom{2n}{k+1} (2n - k - 1) \sum_{j=0}^{k} \left( \binom{k+1}{j} s^{k-j+1} [B_{j,k-j+1,1}, st] t^j \right) \\
&+ (-1)^{k+1} \binom{2n}{k+1} \sum_{j=0}^{k} \left( \binom{k+1}{j} s^{k-j+1} [B_{j,k-j+1,1}, u] t^j \right) \\
&= (-1)^{k+1} \binom{2n}{k+1} (2n - k - 1) \sum_{j=0}^{k+1} \left( \binom{k+1}{j} s^{k-j+1} [B_{j,k-j+1,1}, st] t^j \right) \\
&+ (-1)^{k+1} \binom{2n}{k+1} \sum_{j=0}^{k+1} s^{k-j+1} [B_{j,k-j+1,1}, u] t^j \\
\end{align*}

Accordingly equation (10) becomes

\begin{align*}
&(-1)^{k+1} \binom{2n}{k+1} (2n - k - 1) \sum_{j=0}^{k+1} \left( \binom{k+1}{j} s^{k-j+1} [B_{j,k-j+1,1}, st] t^j \right) \\
&+ \sum_{i=0}^{k+1} \sum_{j=0}^{i} (-1)^i \binom{2n}{i} \binom{i}{j} s^{i-j} [B_{j,i-j}, u] t^j = 0 \\
\end{align*}

This proves equation (10) for all \( k = 0, 1, \ldots, 2n \). In particular when \( k = 2n \) we get

\begin{align*}
\sum_{i=0}^{2n} \sum_{j=0}^{i} (-1)^i \binom{2n}{i} \binom{i}{j} s^{i-j} [B_{j,i-j}, u] t^j &= 0 \\
\Rightarrow \sum_{j=0}^{2n} \Lambda_j(s,t) u t^j - \sum_{j=0}^{2n} s^j u Y_j(s,t) &= 0 \\
\end{align*}

(11)
∀ s, t, u ∈ R where Λ_j(s, t) = \sum_{i=j-1}^{2n} (-1)^i \binom{2n}{i} \binom{i}{j-1} s^{i-j} B_{j,i-j} and Υ_j(s, t) = \sum_{i=j}^{2n} (-1)^i \binom{2n}{i} \binom{i}{j} B_{j,i} t^{i-j}.

We see that Λ_{2n}(s, t) = (-1)^{2n} B_{2n,0} = \binom{2n}{n} M(s, s, \ldots, s)^2 is independent of t.

We claim that [Λ_j(s, t), s] = 0 ∀ s, t ∈ R and j = 0, 1, \ldots, 2n. We prove this by induction. For j = 2n we have [Λ_{2n}(s, t), s] = [(-1)^{2n} B_{2n,0}, s] = \binom{2n}{n} M(s, s, \ldots, s)^2, s] = \binom{2n}{n} \{M(s, s, \ldots, s) \{M(s, s, \ldots, s), s]\} + [M(s, s, \ldots, s), s]

M(s, s, \ldots, s) = 0 by the given condition. Now, suppose 1 ≤ j ≤ 2n and [Λ_j(s, t), s] = 0 ∀ s, t ∈ R. Put s_1 = s_2 = \ldots = s_j = s, s_{j+1} = s_{j+2} = \ldots = s_{i+1} = t and s_{i+2} = \ldots = s_{2n+1} = st in equation (2), we get

(2n - i)[B_{j,i-1}, s] + j[B_{j-1,i-1}, s] + (i - j + 1)[B_{j,i-1}, t] = 0.

Hence

\[ [Λ_{j-1}(s, t), s] = \sum_{i=j-1}^{2n} (-1)^i \binom{2n}{i} \binom{i}{j-1} s^{i-j+1} [B_{j-1,i-j+1}, s] \]

\[ = \sum_{i=j}^{2n} (-1)^i + 1 \binom{2n}{i} \binom{i}{j-1} s^{i-j+1} [B_{j,i-j}, t] \]

\[ + \sum_{i=j}^{2n-1} (-1)^{i+1} \binom{2n}{i} \binom{i}{j} s^{i-j} B_{j,i-j+1, s} \]

\[ = \sum_{i=j}^{2n} (-1)^{i+1} \binom{2n}{i} \binom{i}{j} s^{i-j+1} [B_{j,i-j}, t] \]

\[ + \sum_{i=j}^{2n-1} (-1)^{i+1} \binom{2n}{i+1} \binom{i+1}{j} s^{i-j+1} [B_{j,i-j+1}, s] \]

\[ = \sum_{i=j}^{2n} (-1)^{i+1} \binom{2n}{i} \binom{i}{j} s^{i-j+1} [B_{j,i-j}, t] \]
Now if we define $\Phi_j : R^{2n} \rightarrow S$ by
$$\Phi_j(s_1, s_2, \ldots, s_{2n}) = \sum_{i=j}^{2n} (-1)^{i-j} \binom{2n}{i} s_1 s_2 \cdots s_{i-j} \times B(s_{i-j+1}, \ldots, s_i, s_{i+1} t, \ldots, s_{2n} t, t, \ldots, t),$$
then $\Lambda_j(s, t)$ is the trace of $\Phi_j$ for each $j$ and $\forall t \in R$ and $[\Phi_j(s, s, \ldots, s), s] = [\Lambda_j(s, t), s] = 0 \forall s \in R$. By theorem (1), $\Lambda_j(s, t)$ is a linear combination of $1, s, \ldots, s^{2n}$ over $C$. Since $R$ is not algebraic of bounded degree $\leq 2n$ over $C$, there exist $s_0 \in R$ such that $1, s_0, s_0^2, \ldots, s_0^{2n}$ are linearly independent over $C$. Hence there exist mappings $\alpha_{i,j} : R \rightarrow C; i, j = 0, 1, \ldots, 2n$ such that $\Lambda_j(s_0, t) = \sum_{i=0}^{2n} \alpha_{i,j}(t)s_0^i \forall t \in R$. Thus from (11) we have
$$\sum_{i=0}^{2n} s_0^i u \left( \sum_{j=0}^{2n} \alpha_{i,j}(t)t^j - \hat{Y}_i(s_0, t) \right) = 0$$
$\forall t, u \in R$. Therefore, $\hat{Y}_i(s_0, t) = \sum_{j=0}^{2n} \alpha_{i,j}(t)t^j \forall t \in R$ and $i = 0, 1, \ldots, 2n$.

In particular, $B(t, t, \ldots, t) = \binom{2n}{n} M(t, t, \ldots, t)^2 = \hat{Y}_{2n}(s_0, t) = \sum_{j=0}^{2n} \alpha_{2n,2n-j}(t)t^{2n-j}$
and so \( M(t, t, \ldots, t)^2 = \sum_{j=0}^{2n} \lambda_j(t) t^{2n-j} \) where \( \lambda_j(t) = \frac{1}{\binom{2n}{n}} \alpha_{2n, 2n-j}(t) \). For each \( i = 0, 1, \ldots, 2n, \alpha_{i, 2n}(t) \) are independent of \( t \) since \( \Lambda_{2n}(s_0, t) \) are independent of \( t \). This implies that \( \lambda_0(t) = (-1)^{2n} \alpha_{2n, 2n}(t) \) is independent of \( t \). Now \( \Lambda_j(s_0, t) = \sum_{i=j}^{2n} (-1)^i \binom{2n}{i} \binom{i}{j} s_0^{i-j} B(s_0, \ldots, s_0, t) \) is the trace of an \((2n - j)\)-additive mapping treated as mapping in \( t \). Hence \( \Lambda_j(s_0, t) \) satisfies the two condition of Lemma (2). Also since \( 1, s_0, s_0^2, \ldots, s_0^{2n} \) are linearly independent over \( C \) and \( \Lambda_j(s_0, t) = \sum_{i=j}^{2n} \alpha_{i, j}(t) s_0^i \) so each \( \alpha_{i, j} \) satisfies both the desired conditions and hence is the trace of some \((2n - j)\)-additive mapping. In particular, each \( \lambda_j(t) = \frac{(-1)^{2n}}{\binom{2n}{n}} \alpha_{2n, 2n-j} \) is the trace of some \( j \)-additive mapping for every \( j = 1, 2, \ldots, 2n \). Hence the theorem.

References


