A complete characterization of the unit group $U(Z_pQ_8)$ of the group algebra $Z_pQ_8$ of quaternion group over the field $Z_p$ is obtained. Presentation in terms of generators and relators has been given.
1 Introduction

The unit group of the integral group ring $\mathbb{Z}Q_8$ has already been studied. The unit group contains trivial units only in case of $\mathbb{Z}Q_8$. No attempts however have been made to study the unit group of group algebra $\mathbb{Z}_pQ_8$, where $\mathbb{Z}_p$ denotes the field of integers modulo a prime $p$. In this paper we give presentations in terms of generators and relators of the unit group for every $p$.

We shall need the following results:

**Theorem 1.1.** [Maschke] Let $G$ be a group and $R$ a ring with identity. Then, the group ring $RG$ is semisimple if and only if the following conditions hold.

1. $R$ is a semisimple ring.
2. $G$ is finite.
3. $O(G)$ is invertible in $R$.

**Theorem 1.2.** Let $G$ be a finite group and let $K$ be a field such that $\text{char } K \nmid O(G)$. Then $KG$ is a direct sum of simple components of $KG$ where each simple component is isomorphic to a full matrix ring of the form $M_{n_i}(D_i)$, where $D_i$ is a division ring containing an isomorphic copy of $K$ in its centre.

**Proposition 1.3.** Let $G$ be a finite group and $K$ an algebraically closed field such that $\text{char } K \nmid O(G)$. Then, the number of simple components of $KG$ is equal to the number of conjugacy classes of $G$.

**Definition 1.4.** For a subgroup $H \in S(G)$, where $S(G)$ is the set of all subgroups of $G$, we shall denote by $\Delta(G, H)$ the left ideal of $RG$ generated by the set $\{h - 1 : h \in H\}$. That is,

$$\Delta(G, H) = \left\{ \sum_{h \in H} \alpha_h (h - 1) : \alpha_h \in RG \right\}.$$

**Theorem 1.5.** Let $RG$ be a semisimple group algebra. If $G'$ denotes the commutator subgroup of $G$, then we can write

$$RG \simeq R(G/G') \oplus \Delta(G, G'),$$

where $R(G/G')$ is the sum of all commutative simple components of $RG$ and $\Delta(G, G')$, is the sum of all the others.
2 The Unit Group of $\mathbb{Z}_2Q_8$

Theorem 2.1. Let $Q_8 = \langle x, y | x^4 = y^4 = 1, x^2 = y^2, xy = x^{-1}y \rangle$. Then the unit group $U(\mathbb{Z}_2Q_8)$ of $\mathbb{Z}_2Q_8$ can be presented as follows:

$$G = U(\mathbb{Z}_2Q_8) = \langle a, b, c, d | a^4 = b^4 = c^4 = d^2 = 1, a^2b^2, c^2, d \in Z(G), ba = a^{-1}b, ca = a^{-1}c, (bc)^2 = a^2, (bc)^4 = 1 \rangle,$$

where $a = x, b = 1 + x + y, c = x + x^3 + y + xy$ for $x$ and $y$ generators of $Q_8$. Further the order of $G$ is 128.

Proof. Observe that the total number of elements in $\mathbb{Z}_2Q_8$ is 256 and the number of elements of even length 128.

Any element of even length in $\mathbb{Z}_2Q_8$ cannot be a unit, since any such element belongs to the augmentation ideal $\omega(\mathbb{Z}_2Q_8)$. Hence $O(U(\mathbb{Z}_2Q_8)) \leq 128$ as elements of even length are 128.

Let $G = \langle a, b, c, d | a^4 = b^4 = c^4 = d^2 = 1, a^2b^2, c^2, d \in Z(G), ba = a^{-1}b, ca = a^{-1}c, (bc)^2 = a^2, (bc)^4 = 1 \rangle$, where $a = x, b = 1 + x + y, c = x + x^3 + y + xy$ for $x$ and $y$ generators of $Q_8$. If $H_1 = \langle a, b | a^4 = b^4 = 1, b^2, a^2 \in Z(G), ba = a^3b \rangle$, then $O(H_1) = 16$, since the canonical form of the elements in $H_1$ is $a^ib^j$ for $0 \leq i \leq 3; 0 \leq j \leq 3$. The elements of $H_1$ can be listed as 

$$\{1, x, x^2, x^3, (1 + x + y), (1 + xy + x^3y), (1 + x^2 + xy + x^2y + x^3y), (x + x^2 + xy + x^3y), (x + y + x^2y), (1 + x + y + x^2y + x^3y), (x^2 + x^3 + x^2y), (x^2 + xy + x^3y), (x + x^2 + y + xy + x^3y), (1 + x^3 + x^2y), (x^2 + x^3 + y + xy + x^2y) \}.$$

Further if $H_2 = \langle c, d | c^4 = d^4 = 1, cd = dc \rangle$, then $O(H_2) = 8$, since the canonical form of the elements in $H_2$ is $c^id^j$ for $0 \leq i \leq 3; 0 \leq j \leq 1$. The elements of $H_2$ can be listed as 

$$\{(1, (x + x^2 + x^3 + y + xy), (1 + x + x^3), (x^2 + x^2y + x^3y), (x + x^2 + x^3 + y + xy + x^2y + x^3y), (1 + x^2 + x^3y), (x^2 + y + xy + x^2y + x^3y), (1 + x + x^3 + y + xy) \}.$$

Now $O(H_1H_2) = O(H_1)O(H_2)$. But as we see from above $H_1 \cap H_2 = 1$, therefore $O(H_1H_2) = 128$. This implies $O(G) \geq O(H_1H_2) \geq 128$. But order of the unit group is atmost 128. Hence $G = U(\mathbb{Z}_2Q_8)$ and the order of the $G$ is 128.

3 The Unit Group of $\mathbb{Z}_pQ_8$, $p > 2$

Theorem 3.1. If $p > 2$ is a prime then, $\mathbb{Z}_pQ_8 \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times M_2(\mathbb{Z}_p)$. 

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Proof. Let $Q_8 = \langle x, y | x^4 = y^4 = 1, x^2 = y^2, x^2 \in Z(Q_8), yx = x^3y \rangle$. Observe that $Z(Q_8) = Q'_8 = \{1, a^2\}$. So $Q_8/Q'_8 \simeq \{1, \bar{a}, \bar{b}, \bar{ab}\} \simeq C_2 \times C_2$. Thus $Z_p(Q_8/Q'_8) = Z_p(C_2 \times C_2) \cong Z_p \times Z_p \times Z_p \times Z_p \times Z_p$.

Since $p \not\mid O(Q_8)$, by Maschke's theorem $Z_p Q_8$ is a semi-simple Artinian algebra over $Z_p$. Also it is known that finite division rings are fields. Now by Theorem 1.4, $Z_p Q_8 = Z_p(Q_8/Q'_8) \oplus \Delta(Q_8, Q'_8)$.

Since $|\Delta(Q_8, Q'_8) : Z_p| = 4$, we get that $\Delta(Q_8, Q'_8)$ is either isomorphic to $M_2(Z_p)$ or to a non-commutative division ring of dimension 4 over $Z_p$. Further $\Delta(Q_8, Q'_8) = Z_p Q_8 (1 - e_{Q'_8}) = Z_p Q_8 \left( \frac{1}{2}(1 - a^2) \right)$ since $1 - e_{Q'_8} = \frac{1}{2}(1 - a^2)$.

Thus $\Delta(Q_8, Q'_8)$ contain the elements $\{\frac{1}{2}(1 - a^2), \frac{a}{2}(1 - a^2), \frac{b}{2}(1 - a^2), \frac{ab}{2}(1 - a^2)\}$. An elementary matrix basis for $Z_p Q_8 f$, where $f = 1 - e_{Q'_8}$ over $Z_p$ is given by the elements

- $e_{11} = \frac{1+b}{2} f$, $e_{12} = \frac{1-b}{2} f$,
- $e_{21} = \frac{1+b}{2} af$, $e_{22} = \frac{1-b}{2} af$.

Note that a typical element of $Z_p Q_8 f$ can be written as $(a_0 + a_1 a + a_2 b + a_3 ab) f$ with $a_0, a_1, a_2, a_3 \in Z_p$ and $f = e_{11} + e_{12}$, $af = e_{21} + e_{22}$, $bf = e_{11} - e_{12}$, $abf = e_{21} - e_{22}$.

Hence, $(a_0 + a_1 a + a_2 b + a_3 ab) f$ is represented by the matrix

\[
\begin{pmatrix}
0 & a_0 & a_1 & a_2 & a_3 \\
0 & 0 & 0 & -a_2 & -a_3 \\
\end{pmatrix}
\]

This is a matrix of the form $\begin{pmatrix} r & s \\ t & u \end{pmatrix}$, where $r, s, t, u \in Z_p$. Conversely, it is easy to see that any matrix of this type can be obtained by a suitable choice of $a_0, a_1, a_2, a_3$. Hence

\[
\Delta(Q_8, Q'_8) = Z_p Q_8 (1 - e_{Q'_8}) \cong \left\{ \begin{pmatrix} r & s \\ t & u \end{pmatrix} : r, s, t, u \in Z_p \right\}.
\]

Thus

\[
Z_p Q_8 \cong Z_p \times Z_p \times Z_p \times Z_p \times M_2(Z_p).
\]

Theorem 3.2. The unit group of $Z_p Q_8$ is given by

\[
U(Z_p Q_8) \cong C_{p-1} \times C_{p-1} \times C_{p-1} \times C_{p-1} \times GL_2(Z_p),
\]
where $GL_2(\mathbb{Z}_p)$ denotes the group of $2 \times 2$ invertible matrices over the field $\mathbb{Z}_p$ and $C_{p-1}$ denotes the cyclic group of order $p - 1$. In particular, the order of $U(\mathbb{Z}_p Q_8)$ is $(p - 1)^5 p(p + 1)$.

Proof. From Theorem 3.1 it follows that $U(\mathbb{Z}_p Q_8) \cong C_{p-1} \times C_{p-1} \times C_{p-1} \times C_{p-1} \times GL_2(\mathbb{Z}_p)$. Since the order of $GL_2(\mathbb{Z}_p)$ is $(p^2 - 1)(p^2 - p)$, we get that the order of $U(\mathbb{Z}_p Q_8)$ is $(p - 1)^5 p(p + 1)$.

References


