A Note on Groups of Finite Weight

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Abstract. The weight, \( \omega(G) \), of a group \( G \) is the least cardinality of a subset \( X \) of \( G \) such that \( G = \langle X \rangle_N \). In this paper, a study of groups of finite weight has been carried out. Weight of nilpotent and solvable groups have been studied. Some classes of groups, for which conditions of finite weight and finitely generated are equivalent, have been discussed. Weight of wreath product of certain classes of groups have also been obtained.

Keywords: Normal closure; Weight of a group; Wreath product.

Let \( G \) be a group and \( X \subseteq G \). Denote the normal closure of \( X \) in \( G \) by \( \langle X \rangle_N \). Thus \( \langle X \rangle_N = \langle x^g = g^{-1}xg \mid x \in X, g \in G \rangle \). A group \( G \) is said to have finite weight if there exists a finite subset \( X \) of \( G \) such that \( G = \langle X \rangle_N \). The weight \( \omega(G) = n < \infty \) if \( n \) is the least positive integer such that \( G = \langle X \rangle_N \) where \( X \) has \( n \) distinct elements of \( G \). For a normal subgroup \( H \) of \( G \), we define \( G \)-weight of \( H \), \( \omega_G(H) \), as the least positive integer \( n \) such that \( \exists \) a subset \( Y \subseteq H \) having \( n \) distinct elements and \( H = \langle y^g \mid y \in Y, g \in G \rangle \).

P. Kutzko [1] studied groups of finite weight and proved that If \( G \) is a group of finite weight and the lattice of normal subgroups of \( G \) contained in the commutator subgroup \( G' \) satisfies the minimum condition then \( \omega(G) = \omega(G/G') \).

A.H. Rhemtulla [3] carried out the study further and proved the following stronger theorem. The result by Kutzko [1] follows as a special case of this.

If \( N \) is a normal subgroup of a group \( G \) and \( G \) acts on \( N \) by conjugation, then \( d_G(N) = d_G(N/N') \) provided \( d_G(N) \) is finite and \( N \) has the following property:

There does not exist an infinite descending series of \( G \)-subgroups \( N' = C_0 > \)
$C_1 > ...$ with each $C_i/C_{i+1}$ perfect.

Note that $d_G(N) = w_G(N)$ and $d_G(G) = w(G)$.

Further, since a solvable group cannot have perfect non trivial factors, by [3], we have for a solvable group $G$ of finite weight, $w(G) = w(G/G')$. Also if $G$ is a nilpotent group and $H$ is a subgroup of $G$ such that $G = HG'$ then $G = H$. Hence as a consequence, we get the following theorem.

**Theorem 1.** Let $G$ be a solvable group of finite weight. Then $w(G) = w(G/G')$. Further, if $G$ is a nilpotent group then $w(G) < \infty$ if and only if $G$ is finitely generated. In particular, the nilpotent group $G$ has weight 1 if and only if $G$ is a cyclic group.

We would like to mention that there exists solvable groups of finite weight which are not finitely generated. Also solvable groups of weight 1 need not be cyclic. Weight of certain wreath products can be derived using results of P.M. Neumann [2]. These will lead to examples of solvable groups of finite weight which are not finitely generated. For definitions, notations and details about wreath product, the reader is referred to P.M. Neumann [2].

**Theorem 2.** [2, p352] Let $A$ and $B$ be any two groups and $G = A \wr B$. If $B$ contains an element $b$ of infinite order, then every element of $A^B$ is a commutator, $(A \wr B)' = A^B B'$ and the normal closure of $(b, f)$ contains $A^B$, where $f$ is any arbitrary but fixed element of $A B$.

Using the above theorem, we get the following result on weight of wreath product of groups. The most important observation about the following result is that there is no condition on the group $A$. The theorem is particularly useful in constructing examples and counterexamples.

**Theorem 3.** Let $A$ be a group and $B$ be a group of finite weight such that $B/B'$ is infinite. Then $w(A \wr B) = w(B)$.

*Proof.* As $w(B) < \infty$ and $B/B'$ is infinite, $B$ is non torsion infinite. Let $w(B) = r$. Then $B = \langle b_1, b_2, ..., b_r \rangle_N$. Also as $B/B'$ infinite, at least one normal generator, say $b_1$, is of infinite order. Then by above theorem, $A^B \subseteq \langle (b_1, f) \rangle_N$, hence $(1, f)$ and so $(b_1, 1)$ lies in $\langle (b_1, f) \rangle_N$. Thus $A \wr B \subseteq \langle (b_1, f), (b_2, 1), ..., (b_r, 1) \rangle_N$. \hfill $\blacksquare$

It is easy to see that $w(S_n) = 1$, for a symmetric group of degree $n$. For dihedral group $D_{2n}$, $w(D_{2n}) = 1$ if $n$ is odd; $w(D_{2n}) = 2$ if $n$ is even. Also $w(D_\infty) = 2$, where $D_\infty$ is the infinite dihedral group. If $G$ is any non Abelian simple group (finite or infinite) then clearly $w(G) = 1$. It is not difficult to see that $w(GL_n(K)) = 1$, $n \geq 1$, $K$ a finite field. It follows from the fact that $GL_n(K)/SL_n(K) \cong K^*$, where $K^* = K \setminus \{0\}$ is cyclic as $K$ is a finite field.
Further, $GL_n(K)' = SL_n(K)$ unless $n = 2$ and $K = \mathbb{Z}_2$. In case $n = 2$ and $K = \mathbb{Z}_2$, we know $GL_2(\mathbb{Z}_2) \cong S_3$.

A group $G$ is said to be an FC- group (Finite Conjugate group) if every $x \in G$ has finitely many conjugates in $G$, that is, $|G : C_G(x)| < \infty$ for every $x \in G$.

Finally, we prove the following theorem for FC groups:

**Theorem 4.** Let $G$ be an FC group of finite weight. Then $G$ is finitely generated and $w(G) = w(G/G')$.

**Proof.** Let $w(G) = n < \infty$. So $G = \langle x_1, x_2, ..., x_n \rangle_N = \langle x_i^g | g \in G, i = 1, 2, ..., n \rangle$.

But as $G$ is an FC group, so each $x_i$ has finitely many conjugates in $G$. Hence $G$ is finitely generated. Let $G = \langle y_1, y_2, ..., y_m \rangle$. Now $Z(G) = \cap_{i=1}^m C_G(y_i)$ and $|G : C_G(y_i)| < \infty$ for each $i$, hence $|G : Z(G)| < \infty$. Hence, by a well known result due to Schur, the commutator subgroup $G'$ is finite. Therefore, by Kutzko’s theorem, we get $w(G) = w(G/G')$. 

**References**

