UNITS IN $\mathbb{Z}_2(C_2 \times D_{\infty})$

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Abstract. In this paper we consider the group algebra $R(C_2 \times D_{\infty})$. It is shown that $R(C_2 \times D_{\infty})$ can be represented by a $4 \times 4$ block circulant matrix. It is also shown that $U(\mathbb{Z}_2(C_2 \times D_{\infty}))$ is infinitely generated.

1. Introduction

Let $RG$ denote the group ring of a group $G$ over a ring $R$, and $U(RG)$ denote the unit group of $RG$. A lot is known about the unit group of group rings of finite groups. For details see [9], [1] and [2]. In this paper we deal with units in the group algebra $R(C_2 \times D_{\infty})$ over a commutative integral domain $R$, by representing it by $4 \times 4$ block circulant matrix. The idea that the group ring $RD_{2n}$ can be written as a block matrix was introduced by Hurley in [6]. Additionally this method was also used by Gildea in [4], [5] to establish the structure of certain unit groups of group algebras. Maciez Mirowicz in [7] studied the group of units $U(RD_{\infty})$ of the group ring of the infinite dihedral group $D_{\infty}$ over a commutative integral domain $R$. He obtained the structure of $U(\mathbb{Z}_2D_{\infty})$. In this paper, we extend his results and obtain some subgroups of the unit group $U(\mathbb{Z}_2(C_2 \times D_{\infty}))$. We have shown that $U(\mathbb{Z}_2(C_2 \times D_{\infty}))$ is not finitely generated.

Let $R$ be a commutative domain with unity. The infinite dihedral group is a two generator group with a known presentation as:

$$D_{\infty} = \langle t, x \mid x^2 = 1, xt = t^{-1}x \rangle.$$
$C_2$ is the cyclic group of order 2 generated by $y$, that is, $C_2 = \langle y \rangle$. Since the canonical form of elements of $D_\infty$ is $t^ix^j$ for some $i \in \mathbb{Z}$ and $0 \leq k \leq 1$ and $y$ commutes with $t$ and $x$ we can write any element $\alpha \in R(C_2 \times D_\infty)$ in the form: $\alpha = (a + bx) + (c + dx)y$, where $a$, $b$, $c$, $d \in RC_\infty$, where $C_\infty$ denotes an infinite cyclic group.

Let $C_\infty = \langle t \rangle$ be an infinite cyclic group generated by $t$ and let $*: RC_\infty \to RC_\infty$ be the involution map of the group ring $RC_\infty$ which comes from the non-trivial automorphism of the group $C_\infty$, that is, $\left( \sum_{i \in \mathbb{Z}} a_it^i \right)^* = \sum_{i \in \mathbb{Z}} a_i t^{-i}$. We can easily get that for any $a \in RC_\infty$ the relation $xa = a^*x$ holds.

2. Units in $\mathbb{Z}_2(C_2 \times D_\infty)$

In this section we obtain some infinitely generated subgroups of the unit group of $\mathbb{Z}_2(C_2 \times D_\infty)$. First, we prove some lemmas.

**Lemma 2.1.** Let $\theta : R(C_2 \times D_\infty) \to M_4(RC_\infty)$ defined by

$$\theta((a + bx) + (c + dx)y) = \begin{pmatrix} a & b & c & d \\ b^* & a^* & d^* & c^* \\ c & d & a & b \\ d^* & c^* & b^* & a^* \end{pmatrix} = \begin{pmatrix} A & B \\ B & A \end{pmatrix},$$

where $A = \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix}$ and $B = \begin{pmatrix} c & d \\ d^* & c^* \end{pmatrix}$. Then $\theta$ is a monomorphism.

**Proof.** Let $\alpha = (a_1 + b_1 x) + (c_1 + d_1 x)y$ and $\beta = (a_2 + b_2 x) + (c_2 + d_2 x)y$. Then $\alpha \beta = (p+q x) + (r+s x)y$, where $p = a_1 a_2 + b_1 b_2^* + c_1 c_2 + d_1 d_2^*$, $q = a_1 b_2 + a_2^* b_1 + c_1 d_2 + c_2^* d_1$, $r = a_1 c_2 + a_2 c_1 + b_1 d_2^* + b_2^* d_1$ and $s = a_1 d_2 + a_2^* d_1 + b_1 c_2 + b_2 c_1$.

Now, $\theta(\alpha \beta) = \begin{pmatrix} p & q & r & s \\ q^* & p^* & s^* & r^* \\ r & s & p & q \\ s^* & r^* & q^* & p^* \end{pmatrix} = \theta(\alpha)\theta(\beta)$

Hence $\theta$ is a homomorphism.

As $\theta((a + bx) + (c + dx)y) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\Rightarrow a = b = c = d = 0$, thus $\theta$ is one-one. Hence the lemma.

**Lemma 2.2.** If $R$ is a commutative domain then $\alpha \in \mathcal{U}(RG) \iff \det(\alpha) \in \mathcal{U}(R)$, where $G$ is $(C_2 \times D_\infty)$. 

Proof. Let $\alpha = \begin{pmatrix} a & b & c & d \\ b^* & a^* & d^* & c^* \\ c & d & a & b \\ d^* & c^* & b^* & a^* \end{pmatrix}$. Then, $\det(\alpha) = (aa^* - bb^*)^2 + (cc^* - dd^*)^2 + 2(abc^*d^* + ab^*c^*d + a^*b^*cd) - 2(aa^*dd^* + bb^*cc^*) - (a^2c^*2 + a^2e^2 + b^2d^*2 + b^2d^2)$. Thus we get, $(\det(\alpha))^* = \det(\alpha)$.

Suppose $\alpha \in \mathcal{U}(RG)$ then there exists $\beta \in RG$ such that $\alpha \beta = 1 \Rightarrow \det(\alpha)\det(\beta) = \det(I) = 1$. Thus $\det(\alpha) \in \mathcal{U}(RC_{\infty})$. But $\mathcal{U}(RC_{\infty}) = \{rt^i \mid i \in \mathbb{Z}, r \in R\}$. Thus we have $\det(\alpha) = rt^i$ for some $i$. But as $(\det(\alpha))^* = \det(\alpha)$, we get $(rt^i)^* = rt^i$. This gives $rt^{-i} = rt^i \Rightarrow i = 0$. Hence $\det(\alpha) = r \in R$

For $0 \neq a = \sum_{i \in \mathbb{Z}} \alpha_it^i \in RC_{\infty}$ we fix:

$$\begin{align*}
\max a & := \max\{i \mid \alpha_i \neq 0\} \\
\min a & := \min\{i \mid \alpha_i \neq 0\} \\
\deg a & := \max a - \min a = \max aa^*
\end{align*}$$

If $\alpha = a + bxy \in R(C_2 \times D_{\infty}) \cong \begin{pmatrix} a & o & o & b \\ 0 & a^* & b^* & 0 \\ 0 & b & a & 0 \\ b^* & 0 & 0 & a^* \end{pmatrix}$ is a non-trivial unit then $a \neq 0$, $b \neq 0$.

Thus $\det(\alpha) = (aa^* - bb^*)^2 \in \mathcal{U}(R)$ from Lemma 2.2. Now $(aa^* - bb^*) \in RC_{\infty}$. But $\mathcal{U}(RC_{\infty}) = \{rt^i \mid i \in \mathbb{Z}, \ r \in R\}$. Thus we have $aa^* - bb^* = rt^i$ for some $i$. But as $(aa^* - bb^*)^* = (aa^* - bb^*)$, we get $(rt^i)^* = rt^i$. This gives $rt^{-i} = rt^i \Rightarrow i = 0$. Therefore, $aa^* - bb^* \in \mathcal{U}(R)$.

Hence we define $\deg\alpha = \max aa^* = \max bb^* = \deg b > 0$. For trivial units, we extend this definition by setting $\deg\alpha := 0$.

Let $\text{sgn}(i)$ denotes the sign of $i$. We consider special non-trivial nilpotent elements in the group ring $R(C_2 \times D_{\infty})$:

$\eta_{ij} = (1 + \text{sgn}(i)t^jxy)t^{\mid i \mid}(1 - \text{sgn}(i)t^jxy) = (-t^{-\mid i \mid} + t^\mid i \mid) + \text{sgn}(i)t^j(t^{-\mid i \mid} - t^\mid i \mid)xy$ for $i(\neq 0), \ j \in \mathbb{Z}$.

Also $\eta_{ij}^2 = 0$ as

$$\begin{align*}
(\eta_{ij})^2 &= (1 + \text{sgn}(i)t^jxy)t^{\mid i \mid}(1 + \text{sgn}(i)t^jxy)(1 + t^jxy)t^{\mid i \mid}(1 \mp t^jxy) \\
&= (1 \mp t^jxy)t^{\mid i \mid}(1 \mp t^jxy) \\
&= 0 \text{ because } (t^jxy)^2 = 1.
\end{align*}$$

For any $r \in R, \ i, \ j \in \mathbb{Z}$, the element $1 + r\eta_{ij}$ is a unit in $R(C_2 \times D_{\infty})$. Also inverse of $1 + r\eta_{ij}$ is $1 - r\eta_{ij}$ because

$$(1 + r\eta_{ij})(1 - r\eta_{ij}) = 1 - r^2(\eta_{ij})^2 = 1.$$
All the units of the above form generate a subgroup of the unit group of $R(C_2 \times D_\infty)$, so let

$$U = \langle 1 + r\eta_{ij} \rangle_{i,j \in \mathbb{Z}, r \in R}$$

For all $k > 0$, $j \in \mathbb{Z}$:

$$V_j^k = \langle 1 + r\eta_{ij} \rangle_{0 \leq i \leq k, r \in R}$$

Obviously, the groups $\{V_j^k\}$ form an ascending system. We set:

$$V_j = \lim_{k \to \infty} V_j^k.$$

Natural inclusions induce homomorphisms from the free products:

$$\phi_k : *_j V_j^k \to U \text{ for } k > 0 \text{ and } \phi = \lim_{k \to \infty} \phi_k : *_j V_j \to U.$$ 

Now we describe the groups $V_j^k$. Without loss of generality, we can take $\text{sgn}(i)$ and $\text{sgn}(l)$ to be +ve. Thus

$$\eta_{ij} \cdot \eta_{j} = (1 + t^i xy)t^l(1 - t^i xy)(1 + t^j xy)t^l(1 - t^j xy)$$

$$= (1 + t^i xy)t^l \cdot 0 \cdot t^l(1 - t^j xy) = 0,$$

therefore the function $\sigma : R^k \to V_j^k$ given by:

$$\sigma(r_1, \ldots, r_k) = 1 + r_1\eta_{ij} + \cdots + r_k\eta_{kj}$$

$$= \left(1 - \sum_{i=0}^{k} r_i t^{-i} + \sum_{i=0}^{k} r_i t^i\right) + t^l \left(\sum_{i=0}^{k} r_i t^{-i} - \sum_{i=0}^{k} r_i t^i\right) xy$$

$\sigma$ is an isomorphism from the additive group of $R^k$ onto the multiplicative group $V_j^k$. Therefore we obtain isomorphisms $V_j^k \to R^k$ and $V_j \cong \oplus_{\infty > 0} R$.

**Lemma 2.3.** Let $k > 0$ and let $w \in *_{j \in \mathbb{Z}} V_j^k$ be a non-empty reduced word with the last letter $g$ (i.e., $l(wg^{-1}) < l(w)$), where $l$ denotes the length of the word. If $\phi_k(w) = a + bxy \in U \subseteq \mathcal{U}(\mathbb{Z}_2(C_2 \times D_\infty))$, then:

(i) $\deg \phi_kw > 0$ (in particular $\phi_k$ is a monomorphism)

(ii) $g \in V_j^k \Leftrightarrow \max (t^{-j}b + a) < \max\{\max a, \max t^{-j}b\}$ or $\min (t^{-j}b + a) > \min\{\min a, \min t^{-j}b\}$.

**Proof.** We will prove the result by induction on the length of the word $w$. Let $l(w) = 1$. So, $w \in V_j^k$ for some $j$ and so we can write $w$ as:

$$w = \left(1 + \sum_{i=0}^{k} r_i t^{-i} + \sum_{i=0}^{k} r_i t^i\right) + t^l \left(\sum_{i=0}^{k} r_i t^{-i} + \sum_{i=0}^{k} r_i t^i\right) xy.$$ 

Now, $\phi_k(w) = w \neq 1$ as $w$ is non-empty, hence we have $r_i \neq 0$ for some $1 \leq i \leq k$. Thus, $\deg \phi_k(w) \geq 2i > 0$.

Also, when $l(w) = 1$ then $g = w \in V_j^k$ for some fix $j$. Then $a = \left(1 + \sum_{i=0}^{k} r_i t^{-i} + \sum_{i=0}^{k} r_i t^i\right)$ and $b = t^l \left(\sum_{i=0}^{k} r_i t^{-i} + \sum_{i=0}^{k} r_i t^i\right)$. This implies that

$$\max (t^{-j}b + a) = 0 < \max\{\max a, \max t^{-j}b\}.$$
Hence the result is true for length 1.

For \( c + dxy \in V_j^k \) we obtain some observations as follows:

1. \( c = c^* \), because
\[
c = \sum_{i=1}^{k} r_i t^i + 1 + \sum_{i=1}^{k} r_i t^{-i} \quad \text{and} \quad c^* = \sum_{i=1}^{k} r_i t^i + 1 + \sum_{i=1}^{k} r_i t^{-i},
\]
thus we have, \( c + c^* = 2 \) which implies \( c = c^* \) as char of \( R = \mathbb{Z}_2 \) is 2.

2. \( d^* = dt^{-2j} \), because
\[
d = \left( \sum_{i=1}^{k} r_i t^{-i} + \sum_{i=1}^{k} r_i t^i \right) t^j \quad \text{and} \quad d^* = \left( \sum_{i=1}^{k} r_i t^i + \sum_{i=1}^{k} r_i t^{-i} \right) t^{-j},
\]
thus we have, \( dt^{-j} = d^* t^j \) which implies \( d^* = dt^{-2j} \).

3. \( c + t^{-j}d = 1 \).

Now let us assume that the lemma holds for words of length \( n \geq 1 \). Suppose \( l(w) = n + 1 \) implies \( w = v \cdot g \) where \( l(v) = n \) and \( g \in V_j^k \).

Let \( \phi_k(v) = p + qxy \) and \( g = c + dxy \). So
\[
\phi_k(w) = (p + qxy)(c + dxy) = (pc + qd^*) + (pd + qc^*)xy = a + bxy (\text{say}).
\]

Last word of \( v \) does not belong to \( V_j^k \) and by induction, we obtain the following inequalities:
\[
\begin{align*}
\max (t^{-j}q + p) & \geq \max \{ \max p, \max t^{-j}q \} \quad \ldots \quad (1) \\
\min (t^{-j}q + p) & \leq \min \{ \min p, \min t^{-j}q \} \quad \ldots \quad (2)
\end{align*}
\]
We get
\[
a = pc + qd^* = pc + t^{-2j}q d = pc - t^{-j}q(1 + c) = c(p + t^{-j}q) + t^{-j}q
\]
From (1) it follows that
\[
\max (c(p + t^{-j}q)) = \max c + \max (p + t^{-j}q) \geq \max c + \max t^{-j}q
\]
which implies that
\[
\max a = \max (c(p + t^{-j}q) + t^{-j}q) = \max (c(p + t^{-j}q))
\]
\[
> \max t^{-j}q
\]
Similarly, using (1) we get
\[
\max a = \max c(p + t^{-j}q) > \max p \quad \ldots \quad (4)
\]
By applying similar calculations and replacing \( \max \) by \( \min \), we can obtain \( \min a < \min p \). Thus
\( \deg \phi_k(w) = \deg a = \max a - \min a > \max p - \min p = \deg \phi_k(v) > 0 \). Which completes the induction for (i).

Now, we will prove (ii) part. Let \( g \in V_j^k \), then by using the above mentioned observations we have,
\[
t^{-j}b + a = t^{-j}(pd + qc^*) + (pc + qd^*) = t^{-j}pd + t^{-j}qc + pc + t^{-2j}qd
\]
\[
= (p + t^{-j}q)(c + t^{-j}d) = p + t^{-j}q \quad \text{since} \quad c + t^{-j}d = 1.
\]
Therefore, \( \max (t^{-j}b + a) = \max (p - t^{-j}q) \leq \max \{ \max p, \max t^{-j}q \} \). But \( \max p < \max a \).

Thus, we get \( \max t^{-j}b = \max (p + t^{-j}q + a) > \max t^{-j}q \) by using (3) and (4). So
\[
\max (t^{-j}b + a) \leq \max \{ \max p, \max t^{-j}q \} < \max \{ \max a, \max t^{-j}b \}.
\]
By replacing \( \max \) by \( \min \) and applying the same type of calculations we can easily get that:
Similarly, since in $\mathbb{Z}$

Thus $t \eta 

This completes the proof of the theorem.

Converse of (ii). Suppose $\max(t^{-j}b + a) < \max\{\min a, \max t^{-j}b\}$
or $\min(t^{-j}b + a) > \min\{\min a, \min t^{-j}b\}$.

To show $g \in V^k_j$.

Suppose $g \in V^k_j$ then if $\max(t^{-l}b + a) < \max\{\max a, \max t^{-l}b\}$ for $j, l \in \mathbb{Z}$ or $\min(t^{-l}b + a) > \min\{\min a, \min t^{-l}b\}$. Also $\max(t^{-j}b + a) < \max\{\max a, \max t^{-j}b\}$ then $\max a = \max t^{-j}b$ and hence $\max(t^{-j}b + a) < \max t^{-j}b = j + \max b$. Therefore if $\max(t^{-j}b + a) < \max\{\max a, \max t^{-j}b\}$ and $\max(t^{-l}b + a) < \max\{\max a, \max t^{-l}b\}$ then

\[
\max(t^{-j}b + a + t^{-l}b + a) \leq \max\{\max(a + t^{-j}b), \max(a + t^{-l}b)\} < \max\{\max b - j, \max b - l\}.
\]

But $\max(t^{-j}b + a + t^{-l}b + a) = \max(t^{-j}b + t^{-l}b) = \max b + \max(t^{-j} + t^{-l})$.

So when $t^{-j} - t^{-l} \neq 0$, we have $\max(t^{-j}b + t^{-l}b) = \max\{\max b - j, \max b - l\}$ which gives a contradiction. Therefore $t^{-j} - t^{-l} = 0$, i.e., $j = l$ and hence $g = a + bx \in V^k_j$ for some $j$.

Similarly we can prove the converse if $\min(t^{-l}b + a) > \min\{\min a, \min t^{-l}b\}$ and $\min(t^{-l}b + a) > \min\{\min a, \min t^{-l}b\}$.

\[\square\]

Theorem 2.4. Let $G = \langle U, D \rangle$, where $U = \langle 1 + r\eta_j \rangle_{i,j \in \mathbb{Z}, r \in R}$ and $D \cong (C_2 \times D_\infty) \times U(\mathbb{Z}_2)$, i.e., $D$ denote the group of trivial units of $\mathbb{Z}_2(C_2 \times D_\infty)$. Then:

(i) $U \cong *_{j \in \mathbb{Z}}V^j_j \cong *_{\mathbb{Z} \oplus \mathbb{N}} R^+$, where $R^+$ denotes the additive group of the ring $R = \mathbb{Z}_2$.

(ii) $G = UD$.

Proof. (i) To prove (i), we have to show that the homomorphism $\phi = \lim_k \phi_k : *_{j}V^j_j \to U$ is an isomorphism. The mapping $\phi$ is on to because each generator $1 + r\eta_j$ lies in the image of $\phi$. Further, $\phi$ is one-one because for $1 \neq w \in *_{j}V^j_j$ there exists $k \in \mathbb{N}$ such that $w \in *_{j}V^j_j$ and by above lemma $\phi(w) = \phi_k(w) \neq 1$. This proves part (i).

(ii) In order to prove (ii) it is enough to show that:

1. $U \cap D = \{1\}$ 2. $U$ is a normal subgroup of $G$

1. If $1 \neq \alpha \in U$ then by previous lemma $\deg \alpha > 0$ so $\alpha \not\in D$. Thus $U \cap D = \{1\}$.

2. $D \cong (C_2 \times D_\infty) \times U(\mathbb{Z}_2)$, where $C_2 = \langle y \rangle$, $y$, $U(\mathbb{Z}_2)$ are contained in center of $\mathbb{Z}_2(C_2 \times D_\infty)$, therefore it is sufficient to show that $tUt^{-1} \subseteq U$ and $xUx^{-1} \subseteq U$.

Since in $\mathbb{Z}_2(C_2 \times D_\infty)$, $\eta_{ij} = (t^{-i} + t^i) + t^j(t^{-i} + t^i)xy$.

Thus $t\eta_{ij}t^{-1} = (t^{-i} + t^i) + t^{i+2}(t^{-i} + t^i)xy$.

Similarly, $x\eta_{ij}x^{-1} = (t^{-i} + t^i) + t^{-j}(t^{-i} + t^i)xy$ and hence

\[
tUt^{-1} = t(1 + rt\eta_{ij}t^{-1}) = 1 + r\eta_{k(i+2)} \in U
\]

\[
xUx^{-1} = x(1 + rt\eta_{ij})x^{-1} = 1 + r\eta_{l(-j)} \in U.
\]

This completes the proof of the theorem.

\[\square\]
Remark 2.5. By Lemma 2.3, $1 \neq \alpha \in \text{im } \phi_k$ it implies that $\deg \alpha > 0$ so $\text{im } \phi_k \cap D = \{1\}$. Also by the above Theorem 2.4, $\eta \in \text{im } \phi_k$ implies that $\deg \alpha > 0$. Also by the above Theorem 2.4, $\eta \in \text{im } \phi_k$ is a normal subgroup of $\langle \text{im } \phi_k, D \rangle$. Thus $\langle \text{im } \phi_k, D \rangle = \text{im } \phi_k D$.

Proposition 2.6. $U$ and $G$ are infinitely generated subgroups of unit group of $\mathbb{Z}_2(C_2 \times D_{\infty})$.

Proof. If $\alpha_1, \alpha_2, \ldots, \alpha_n \in U$ then there exists $k \in \mathbb{N}$ such that $\alpha_1 \alpha_2 \cdots \alpha_n \in \phi(*jV_j^k)$. But $1 + \eta_{(k+1)}j \notin \text{im } \phi_k$ because $\phi_{k+1}$ is a monomorphism. Therefore, $\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle \notin U$. Similarly, if $\beta_1, \beta_2, \ldots, \beta_n \in G$ then by above theorem there exists $k \in \mathbb{N}$ such that $\beta_1 \beta_2 \cdots \beta_n \in \phi_k D$. But by above remark, $1 + \eta_{(k+1)}j \notin \text{im } \phi_k D$ because $\phi_{k+1}$ is a monomorphism. Therefore, $\langle \beta_1, \beta_2, \ldots, \beta_n \rangle \notin G$. □

Theorem 2.7. Any $\alpha = a + bxy \in U(\mathbb{Z}_2(C_2 \times D_{\infty}))$ is in $G$ (as above defined).

Proof. $\alpha = a + bxy$. If $\deg \alpha = 0$ then $\alpha$ is a trivial unit. Hence we assume that $\deg \alpha > 0$.

Let $j = \max b - \max a$, and $k = \min \{\min(a + t^{-j}b) - \min a, \max a - \max(a + t^{-j}b)\}$.

Observe that $aa^* - bb^* = 1$ this implies that $aa^* \neq bb^*$ and hence $a \neq t^{-j}b$ since if $a = t^{-j}b$ then $aa^* = bb^*$. Also $k \geq 1$ because $\max aa^* = \max bb^*$ and $\deg \alpha = \deg a = \deg b$ this gives $\max a - \min a = \max b - \min b$ and thus $j = \max a - \max b = \min a - \min b$. Hence $\min(a + t^{-j}b) = \min a - \min b + \min b = \min a$, so $\min(a + t^{-j}b) > \min a > 0$. Similarly, $\max a > \max(a + t^{-j}b)$.

$$
\alpha(1 + \eta_{kj}) = (a + bxy)[(1 + t^k + t^{-k}) + t^j(t^k + t^{-k})xy]
= [a + t^k(a + t^{-j}b) + t^{-k}(a + t^{-j}b)]
+ [b(1 + t^k + t^{-k}) + a(t^j(t^k + t^{-k})xy].
$$

Let $h = t^k(a + t^{-j}b) + t^{-k}(a + t^{-j}b)$.

$$
\max h = k + \max(a + t^{-j}b)(a \geq 1)
\leq \max a - \max(a + t^{-j}b) + \max(a + t^{-j}b)
\leq \max a \text{ (by using definition of } k).
$$

Similarly, we can show that $\min h \geq \min a$.

By definition of $k$ either $\max h = \max a$ or $\min h = \min a$.

Case 1 If $\max h = \max a$. Then $\max(a + h) < \max a$.

$$
\deg(\alpha(1 + \eta_{kj})) = \deg(a + h) = \max(a + h) - \min(a + h)
< \max a - \min a = \deg \alpha.
$$

Similarly, if $\min h = \min a$, we can get $\deg(\alpha(1 + \eta_{kj})) < \min a$.

Hence by induction we can show that $\alpha \in G$.

□

Corollary 2.8. Every unit in $U(\mathbb{Z}_2(C_2 \times D_{\infty}))$ of the form $ax + by = (a + bxy)x \in G$.

If $\alpha = a + bx \in R(C_2 \times D_{\infty}) \cong \begin{pmatrix} a & b & 0 \n 0 & 0 & a \n 0 & 0 & a \n \end{pmatrix}$ is a non-trivial unit then $\alpha \neq 0$, $b \neq 0$.

Thus $\det(\alpha) = (aa^* - bb^*)^2 \in U(R)$ from Lemma 2.2. Now $(aa^* - bb^*) \in RC_{\infty}$. Therefore, $aa^* - bb^* \in U(R)$ as we have shown earlier.

Hence $\deg \alpha = \max aa^* = \max bb^* = \deg b > 0$. 

By considering special non-trivial nilpotent elements in the group ring $R(C_2 \times D_\infty)$ of the form:

$$\delta_{ij} = (1 + \text{sgn}(i)t^{|i|}(1 - \text{sgn}(i)t^{|j|})$$

$$= (-t^{-|i|} + t^{|i|}) + \text{sgn}(i)t^{|i|}(t^{-|i|} - t^{|i|}x)$$

for $i(\neq 0), j \in \mathbb{Z}$.

For any $r \in R$, $i, j \in \mathbb{Z}$, the element $1 + r\delta_{ij}$ is a unit in $R(C_2 \times D_\infty)$. Also inverse of $1 + r\delta_{ij}$ is $1 - r\delta_{ij}$ because

$$(1 + r\delta_{ij})(1 - r\delta_{ij}) = 1 - r^2(\delta_{ij})^2 = 1.$$  

All the units of the above form generate a subgroup of the unit group of $R(C_2 \times D_\infty)$. Let

$$V = \langle 1 + r\delta_{ij} \rangle_{i,j \in \mathbb{Z}, r \in R}$$

For all $k > 0$, $j \in \mathbb{Z}$, define

$$V^k_j = \langle 1 + r\delta_{ij} \rangle_{i < k, r \in R}$$

Obviously, the groups $V^1_j \subseteq V^2_j \subseteq \cdots$ is an ascending chain. We set:

$$V_j = \lim_k V^k_j$$

Thus, we can get following results, that follows from [7].

**Theorem 2.9** ([7]). Let $G' = \langle V, D \rangle$, where $V = \langle 1 + r\delta_{ij} \rangle_{i,j \in \mathbb{Z}, r \in R}$ and $D \cong (C_2 \times D_\infty) \times \mathcal{U}(\mathbb{Z}_2)$, i.e., $D$ denotes the group of trivial units of $\mathbb{Z}_2(C_2 \times D_\infty)$. Then:

(i) $V \cong *_{j \in \mathbb{Z}} V_j \cong *_{\mathbb{Z} \oplus \mathbb{N}} R^+$, where $R^+$ denotes the additive group of the ring $R$.

(ii) $G' = VD$.

**Corollary 2.10.** $V$ and $G'$ are infinitely generated subgroups of $\mathcal{U}(\mathbb{Z}_2(C_2 \times D_\infty))$.

**Theorem 2.11.** Any $\alpha = a + bx \in \mathcal{U}(\mathbb{Z}_2(C_2 \times D_\infty))$ is in $G'$ (as above defined).

**Corollary 2.12.** Every unit in $\mathcal{U}(\mathbb{Z}_2(C_2 \times D_\infty))$ of the form $ay + bx = (ay + bx)y \in G'$.

**References**


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