Lie Centrally Metabelian Group Rings in Characteristic 3

B. Külshammer

Department of Mathematics, University of Augsburg, D-86135 Augsburg, Germany

and

R. K. Sharma

Department of Mathematics, Indian Institute of Technology, Kharagpur (W. B.)-721302, India

Communicated by Walter Feit

Received January 12, 1995

We classify Lie centrally metabelian group algebras over fields of characteristic 3. © 1996 Academic Press, Inc.

Let \( F \) be a field of characteristic \( p \geq 0 \), and let \( L \) be a Lie algebra over \( F \). For subsets \( X, Y \) of \( L \), we denote by \([X, Y]\) the \( F \)-subspace of \( L \) spanned by all elements of the type \([x, y]\) with \( x \in X \) and \( y \in Y \). Then \( L' := [L, L] \) is the derived Lie algebra of \( L \), \( L'' := (L')' \) is the derived Lie algebra of \( L' \), and \( L \) is called centrally metabelian if \([L'', L] = 0\).

For an associative, unitary algebra \( A \) over \( F \), we denote by \( \mathcal{L}(A) \) the Lie algebra which has the same underlying vector space as \( A \) and which satisfies \([a, b] := ab - ba\) for \( a, b \in A \). We call \( A \) Lie centrally metabelian, if \( \mathcal{L}(A) \) is centrally metabelian.

Let \( G \) be a group. For subsets \( X, Y \) of \( G \), we denote by \((X, Y)\) the subgroup of \( G \) generated by all elements of the form \((x, y) := xyx^{-1}y^{-1}\) with \( x \in X \) and \( y \in Y \). Then \( G' := (G, G) \) is the derived subgroup of \( G \), \( G'' := (G')' \) is the derived subgroup of \( G' \), and \( G \) is called centrally metabelian if \((G'', G) = 1\). Moreover, the lower central series of \( G \) is defined by \( \gamma_1(G) := G \) and \( \gamma_{n+1}(G) := [\gamma_n(G), G] \) for every positive integer \( n \).

We are interested in the question of when the group ring \( FG \) of \( G \) over \( F \) is Lie centrally metabelian. For the case \( p = 0 \) it is known that \( FG \) is Lie centrally metabelian if and only if \( G \) is abelian. It is proved in [5] that
the same is true if \( p > 3 \). This leaves the (more interesting) cases \( p \in \{2, 3\} \).

If \( p = 3 \) and if \( FG \) is Lie centrally metabelian then it is also proved in [5] that \( G' \) is a finite 3-Engel 3-group of exponent at most 9 and therefore nilpotent of class at most 4. The main purpose of this paper is to solve the case \( p = 3 \) completely by showing:

**Theorem 1.** Let \( F \) be a field of characteristic 3, and \( G \) a group. Then the group algebra \( FG \) of \( G \) over \( F \) is Lie centrally metabelian if and only if \( |G'| \leq 3 \). Further, \( \mathcal{U}(FG) \), the group of units of \( FG \), is centrally metabelian and its derived subgroup is of exponent \( \leq 3 \).

We also improve an earlier theorem for arbitrary Lie solvable rings of [4] as:

**Theorem 2.** Let \( R \) be a Lie solvable ring of derived length \( n \). Then the ideal of \( R \) generated by all elements of the type \([[[x, y], [z, u]], v]; x, y, z, u, v \in R \) is nilpotent of index at most \( \frac{3}{2} (19.10^{n-3} - 1) \). Further, if 2 is invertible in \( R \), then the ideal of \( R \) generated by all elements of the type \([x, y] = xy - yx \) is a nil ideal.

We start by proving the easy half of Theorem 1.

**Lemma 3.** Let \( F \) be a field of characteristic 3, and let \( G \) be a group such that \( |G'| \leq 3 \). Then the group algebra \( FG \) of \( G \) over \( F \) is Lie centrally metabelian, and \( \mathcal{U}(FG) \), the group of units of \( FG \), is centrally metabelian with its derived subgroup of exponent at most 3.

**Proof.** For any group \( H \), let \( \Delta(FH) \) denote the augmentation ideal of \( FH \). Thus \( \Delta(FH) \) is the \( F \)-span of all elements of the form \( h - 1 \) with \( h \in H \). The canonical epimorphism \( \varphi: FG \rightarrow F[G/G'] \) has kernel \( \Delta(FG')FG \); in particular, \( FG/\Delta(FG')FG \equiv F[G/G'] \) and thus, \( \mathcal{Z}(FG) \subseteq \Delta(FG')FG \) and \( \mathcal{Z}(FG)^{'} \subseteq \Delta(FG')FG\Delta(FG')FG = \Delta(FG')^{2}FG \).

We may assume that \( |G'| = 3 \), for otherwise the result is trivial. If \( g \) denotes a generator of \( G' \) then \( \Delta(FG') = (g - 1)FG' \), \( \Delta(FG')FG = (g - 1)FG \), and \( \Delta(FG')^{2}FG = (g - 1)^{2}FG \), where \( (g - 1)^{2} = g^{2} + g + 1 \) is contained in the centre \( \mathcal{Z}FG \) of \( FG \). Thus

\[
[(g - 1)^{2}a, b] = (g - 1)^{2}[a, b] \in (g - 1)^{2}\mathcal{Z}(FG)^{'} \subseteq (g - 1)^{3}FG = 0
\]

for \( a, b \in FG \), so that \( FG \) is Lie centrally metabelian. Theorem 4.1 in [5] shows that

\[
\mathcal{U}(FG)^{''} \subseteq 1 + \mathcal{Z}(FG)^{''}FG \subseteq 1 + (g - 1)^{2}FG = 1 + (g^{2} + g + 1)FG.
\]

If \( x, y \in G \) then \( (y, x) \in G' = \{1, g, g^{2}\} \) and therefore \( (g^{2} + g + 1)(y, x) = g^{2} + g + 1 \). Thus

\[
(g^{2} + g + 1)xy = (g^{2} + g + 1)(y, x)xy = (g^{2} + g + 1)yx = y(g^{2} + g + 1)x.
\]
This shows that \((g^2 + g + 1)FG \subseteq \mathcal{Z}FG\); in particular, \((\mathcal{Z}(FG)', \mathcal{Z}(FG)) = 1\). Finally, \(\mathcal{Z}(FG)' \subseteq 1 + \mathcal{Z}(FG)FG \subseteq 1 + \Delta(\mathcal{Z}FG')FG = 1 + (g - 1)FG\) implies that

\[
(\mathcal{Z}(FG))'^3 \subseteq (1 + (g - 1)FG)^3 = 1 + (g - 1)^3FG = 1.
\]

The proof of the other half of Theorem 1 is more involved. We start by showing that the group algebra of a certain group of order 18 is not Lie centrally metabelian.

**Lemma 4.** Let \(F\) be a field of characteristic 3, let \(N\) be an elementary abelian group of order 9, let \(a\) be the automorphism of \(N\) inverting all elements in \(N\), and let \(G\) be the semidirect product of \(N\) and \(\langle a \rangle\). Then the group algebra \(FG\) is not Lie centrally metabelian.

**Proof.** We write \(N = \langle b, c \rangle\). Then \(G = \langle a, b, c \rangle\) where \(b^3 = c^3 = 1\), \(bc = cb\), \(aba^{-1} = b^{-1}\), and \(aca^{-1} = c^{-1}\). A simple calculation shows that \([a - bab^{-1}, a - cac^{-1}] \notin \mathcal{Z}FG\). Thus \(FG\) is not Lie centrally metabelian.

We now prove a result on the derived Lie algebra of a tensor product of associative algebras. For simplicity of notation, we shall write \(A', A'', \ldots \) instead of \(\mathcal{L}(A), \mathcal{L}(A)'', \ldots \) in the following:

**Proposition 5.** Let \(A\) and \(B\) be two associative, unitary algebras over a field \(F\). Then

\[
\begin{align*}
(i) \quad (A \otimes B)' &= A' \otimes B + A \otimes B', \\
(ii) \quad [A' \otimes B, A' \otimes B] &= A'' \otimes B + (A')^2 \otimes B'; \\
(iii) \quad (A \otimes B)'' &= A'' \otimes B + (A')^2 \otimes B' + [A' \otimes B, A \otimes B'] + A \otimes B'' + A' \otimes (B')^2.
\end{align*}
\]

**Proof.** (i) Since \((A \otimes B)'\) is spanned by elements of the form

\[
[a_1 \otimes b_1, a_2 \otimes b_2] = a_1a_2 \otimes b_1b_2 - a_2a_1 \otimes b_2b_1
\]

\[
= a_1a_2 \otimes b_1b_2 - a_2a_1 \otimes b_1b_2
\]

\[
+ a_2a_1 \otimes b_1b_2 - a_2a_1 \otimes b_2b_1
\]

\[
= [a_1, a_2] \otimes b_1b_2 + a_2a_1 \otimes [b_1, b_2]
\]

with \(a_1, a_2 \in A\), \(b_1, b_2 \in B\), we obtain \((A \otimes B)' \subseteq A' \otimes B + A \otimes B'\).

Conversely, \(A' \otimes B\) is spanned by elements of the form

\[
[a_1, a_2] \otimes b = (a_1a_2 - a_2a_1) \otimes b = a_1a_2 \otimes b - a_2a_1 \otimes b
\]

\[
= (a_1 \otimes 1)(a_2 \otimes b) - (a_2 \otimes b)(a_1 \otimes 1)
\]

\[
= [a_1 \otimes 1, a_2 \otimes b]
\]
with \( a_1, a_2 \in A, b \in B \). Thus \( A' \otimes B \subseteq (A \otimes B)' \) and, similarly, \( A \otimes B' \subseteq (A \otimes B)' \).

(ii) As in the proof of (i), \([A' \otimes B, A' \otimes B]\) is spanned by elements of the form \([a_1 \otimes b_1, a_2 \otimes b_2] = [a_1, a_2] \otimes b_1b_2 + a_2a_1 \otimes [b_1, b_2]\) with \( a_1, a_2 \in A', b_1, b_2 \in B \). Thus \([A' \otimes B, A' \otimes B] \subseteq A'' \otimes B + (A')^2 \otimes B'\).

Conversely \( A'' \otimes B \) is spanned by elements of the form \([a_1 \otimes b_1, a_2 \otimes b_2] = [a_1 \otimes 1, a_2 \otimes b]\) with \( a_1, a_2 \in A', b \in B \). Thus \( A'' \otimes B \subseteq [A' \otimes B, A' \otimes B] \).

Also, \((A')^2 \otimes B'\) is spanned by elements of the form

\[
a_2a_1 \otimes [b_1, b_2] = [a_1 \otimes b_1, a_2 \otimes b_2] - [a_1, a_2] \otimes b_1b_2
\]

with \( a_1, a_2 \in A', b_1, b_2 \in B, a_1 \otimes b_1, a_2 \otimes b_2 \in A' \otimes B, \) and \([a_1, a_2] \otimes b_1b_2 \in A'' \otimes B \subseteq [A' \otimes B, A' \otimes B] \). Thus \((A')^2 \otimes B' \subseteq [A' \otimes B, A' \otimes B] \).

(iii) An argument similar to the one in (ii) shows that

\[
[A \otimes B', A \otimes B'] = A \otimes B'' + A' \otimes (B')^2.
\]

The result follows from this, (i), and (ii).

In the following \( S_n \) will denote the symmetric group of degree \( n \), for any positive integer \( n \). We use Proposition 5 to show that the group rings of certain groups are not Lie centrally metabelian.

**Lemma 6.** Let \( F \) be a field of characteristic 3, let \( X \) be a group, and let \( G := S_3 \times X \). If the group ring \( FG \) is Lie centrally metabelian, then \( X \) is abelian.

**Proof.** We identify \( FG \) with \( FS_3 \otimes FX \) as usual. By Proposition 5,

\[
(FS_3 \otimes FX)^\prime \cong ((FS_3)^\prime)^2 \otimes (FX)^\prime.
\]

We write \( S_3 = \langle g, h \rangle \) where \( g^3 = h^2 = 1 \) and \( hgh^{-1} = g^2 \). Then \( g - g^2 = g - hgh^{-1} \in (FS_3)^\prime \) and \( z := (g - g^2)^2 = 1 + g + g^2 \in \mathcal{Z}FS_3 \setminus \{0\} \). If \( FG \) is Lie centrally metabelian then \( z \otimes (FX)^\prime \) is contained in \( \mathcal{Z}(FS_3 \otimes FX) \) = \( \mathcal{Z}FS_3 \otimes \mathcal{Z}FX \), so \( (FX)^\prime \subseteq \mathcal{Z}FX \). It can be checked that this implies \( X \) to be abelian.

We now embark on the proof of the more difficult half of Theorem 1.

**Lemma 7.** Let \( F \) be a field of characteristic 3, let \( G \) be a group, and suppose that the group ring \( FG \) of \( G \) over \( F \) is Lie centrally metabelian. Then \( |G'| \leq 3 \).

**Proof.** Let \( G \) be a counterexample. We will often make use of the fact that the hypothesis of Lemma 7 carries over to the subgroups and factor groups of \( G \). The results obtained in [5] show that \( G' \) is a finite 3-group.
and that \((g, h)^3 = 1\) for \(g, h \in G\); in particular, \(G'\) cannot be cyclic. This implies, by Burnside's basis theorem, that \(G'/\Phi(G')\) is not cyclic; here \(\Phi(H)\) denotes the Frattini subgroup of a group \(H\). Thus \(G/\Phi(G')\) is also a counterexample. We may therefore replace \(G\) by \(G/\Phi(G')\) and assume that \(G'\) is elementary abelian of order \(3^n > 3\).

Let \(\mathcal{C} := C_G(G')\) denote the centralizer of \(G'\) in \(G\). The action of \(G\) on \(G'\) by conjugation embeds \(G/\mathcal{C}\) into the automorphism group \(\mathcal{A}(G')\) of \(G'\). Note that \(\mathcal{A}(G')\) is isomorphic to the general linear group \(GL_n(3)\) of degree \(n\) over the field \(\mathcal{F}_3\) with 3 elements, in particular \(G/\mathcal{C}\) is abelian. We now try to determine the structure of \(G/\mathcal{C}\) in more detail. (Once the lemma is proved we will know that \(|G/\mathcal{C}| < 2\).)

Suppose first that \(G/\mathcal{C}\) contains an element \(a\mathcal{C}\) of prime order \(p > 3\). By Maschke's theorem, \(\langle a\mathcal{C}\rangle\) acts completely reducibly by conjugation on \(G'\). We write \(G' = M_1 \times M_2 \times \cdots \times M_r\) with irreducible \(\langle a\mathcal{C}\rangle\)-modules \(M_1, M_2, \ldots, M_r\). Since \(\langle a\mathcal{C}\rangle\) does not act trivially on \(G'\), at least one of the \(M_i\) is a nontrivial \(\langle a\mathcal{C}\rangle\)-module. We may assume that \(i = 1\). Then \(|M_1| > 3\), and we set \(H := \langle a, G'\rangle/M_2 \times M_3 \times \cdots \times M_r\). If we denote the images of \(a\) and \(M_1\) in \(H\) by \(\bar{a}\) and \(\bar{M}_1\), respectively, then \(H = \langle \bar{a}, \bar{M}_1 \rangle = \langle \bar{a}\rangle \bar{M}_1\) where \(\bar{a}\) induces an automorphism of order \(p\) on \(\bar{M}_1 \cong M_1\). Since \(\bar{M}_1\) is a nontrivial irreducible \(\langle \bar{a}\rangle\)-module we get \(\bar{M}_1 = (\bar{M}_1, \langle \bar{a}\rangle) \subseteq H'\) (cf. Satz III.13.4 in [2]); in particular, \(H\) is also a counterexample. Thus we may replace \(G\) by \(H\) and therefore assume that \(G = G'\langle a\rangle\) where \(G'\) is a nontrivial irreducible \(\langle a\mathcal{C}\rangle\)-module.

Since \(a^p\) centralizes both \(G'\) and \(a\) we must have \(a^p \in Z_G\), the centre of \(G\). Thus \(\langle a^p\rangle \cap G'\) is an \(\langle a\mathcal{C}\rangle\)-submodule of \(G'\) on which \(\langle a\mathcal{C}\rangle\) acts trivially. Hence \(\langle a^p\rangle \cap G' = 1\). But now \(G/\langle a^p\rangle\) is also a counterexample. We may therefore replace \(G\) by \(G/\langle a^p\rangle\) and assume that \(G\) is the semidirect product of \(G'\) and \(\langle a\rangle\) where \(a\) has order \(p\). However, in this situation, Theorem C in [3] implies that \(\mathcal{L}(FG)^{\prime \prime} \neq 0\), contradicting our hypothesis, \([\mathcal{L}(FG)^{\prime \prime}, \mathcal{L}(FG)] = 0\).

This contradiction means that \(G/\mathcal{C}\) has order \(2^k3^l\), with integers \(k, l\). Next, we assume that \(G/\mathcal{C}\) contains an element of order 4. Then we argue in a similar way, this time choosing \(M_1\) to be an irreducible faithful \(\langle a\mathcal{C}\rangle\)-module. Thus \(|M_1| = 9\). As before we are led to the situation where \(G = \langle a\rangle M_1\) is the semidirect product of \(M_1\) and \(\langle a\rangle\), where \(\langle a\rangle\) has order 4. Thus \(G\) has order 36, and the subgroup \(H\) or order 18 in \(G\) is isomorphic to the group appearing in Lemma 4. Since \(FH\) is not Lie centrally metabelian by Lemma 4, this is a contradiction.

This contradiction shows that \(G/\mathcal{C}\) is abelian of order \(2^k3^l\) with elementary abelian Sylow 2-subgroup; in particular, \(\mathcal{F}_3\) is a splitting field for \(G/\mathcal{C}\). Thus the \(G/\mathcal{C}\)-module \(G'\) has a composition series with composition factors of \(\mathcal{F}_3\)-dimension 1, i.e., of order 3. In particular, \(G\) has a
normal subgroup $N$ contained in $G'$ with $|G':N| = 9$. Thus we can replace $G$ by $G/N$ and therefore assume that $|G'| = 9$.

In this situation the action of $G$ on $G'$, by conjugation, embeds the abelian group $G/\mathcal{C}$ into the subgroup $B$ of $\mathbb{L}_2(3)$ consisting of all elements of the shape

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$  

Note that $B$ is isomorphic to $S_2 \times S_3$. We assume that $G/\mathcal{C}$ contains an element $a\mathcal{C}$ whose image in $B$ is $-1$. Arguing as before, we see that we may assume that $G = \langle a, G' \rangle = \langle a \rangle G'$ is the semidirect product of $G'$ and $\langle a \rangle$ where $a$ has order 2. Thus $G$ is the group appearing in Lemma 4, which is impossible. This contradiction shows that the abelian group $G/\mathcal{C}$ is isomorphic to a subgroup of $B/\langle -1 \rangle \cong S_3$, in particular, $|G/\mathcal{C}| \leq 3$.

Here we stop our first line of attack and prepare for the second.

We can write $G' = \langle (a, b), (c, d) \rangle$ and set $H := \langle a, b, c, d \rangle$. Thus $H' = G'$, so $H$ is also a counterexample. Hence we may replace $G$ by $H$ and therefore assume that $G = \langle a, b, c, d \rangle$; in particular, $G$ is finitely generated. It follows that the first cohomology group $H^1(G/\mathcal{C}, G')$ is finite, a fact we are going to exploit in a moment. Each element in $\mathcal{C}$ acts trivially on both $G/\mathcal{C}$ and $G'$. So the method of Satz 1.17.1 in [2] yields a homomorphism $\mathcal{C} \to H^1(G/\mathcal{C}, G')$ with kernel $G'\mathcal{C}G$. Since $H^1(G/\mathcal{C}, G')$ is a finite 3-group, $\mathcal{C}/G'\mathcal{C}G$ is a finite 3-group as well. Moreover, $|G'\mathcal{C}G : \mathcal{C}G| = |G' : G' \cap \mathcal{C}G|$ divides 9. Thus $\mathcal{C}/\mathcal{C}G$ is a finite 3-group; in particular, $\mathcal{C}G$ is a subgroup of finite index in the finitely generated group $G$. This implies by Satz 1.19.10 in [2] that $\mathcal{C}G$ is a finitely generated abelian group. Suppose that $\mathcal{C}G$ contains an element $z$ such that $\langle z \rangle \cap G' = 1$. Then $G/\langle z \rangle$ is also a counterexample, so we can replace $G$ by $G/\langle z \rangle$. This applies, in particular, whenever $z$ is an element of infinite order in $\mathcal{C}G$. Thus we may assume that $\mathcal{C}G$ is finite. By a similar argument, we may even assume that $\mathcal{C}G$ is a finite 3-group.

Now $\mathcal{C}$ is a finite 3-group, and $|G : \mathcal{C}| \leq 3$; in particular, $G$ is a finite group of order $3^m$ or $2.3^m$ for some integer $m$. We may also assume that our counterexample has least possible order. Then every proper subgroup $H$ of $G$ satisfies $|H'| \leq 3$.

Let us first consider the case when $|G| = 2.3^m$. We denote by $P$ a Sylow 3-subgroup of $G$ and by $a$ an element of order 2 in $G$, so that $G = \langle a \rangle P$ is a semidirect product. If $P$ is abelian then $P = \mathcal{C}(a) \times (P, \langle a \rangle)$, by Satz III.13.4 in [2]. Since $G/(P, \langle a \rangle)$ is abelian we see that $(P, \langle a \rangle) = G'$. But now $G/\mathcal{C}(a)$ is isomorphic to the group of order 18 appearing in Lemma 4, a contradiction. Next we consider the case when $P$ is non abelian, i.e., $|P| = 3$. In this case, $\bar{P} := P/P' = \mathcal{C}(a) \times (\bar{P}, \langle a \rangle)$ by Satz III.13.4 in [2].
Let $\widetilde{G} := G/P'$. Since $\widetilde{G}/(\tilde{P}, \langle a \rangle)$ is abelian, we conclude that $(\tilde{P}, \langle a \rangle) = G'/P'$; in particular, $|\tilde{P}, \langle a \rangle| = 3$. Now $H := \langle a, G' \rangle$ is a group of order 18, and $a$ acts non-trivially on $G'/P'$. If $a$ acts also non-trivially on $P'$, $H$ is isomorphic to the group appearing in Lemma 4, which is impossible. Thus $a$ acts trivially on $P'$, and $H = P' \times S$ for a subgroup $S$ of $H$ isomorphic to $S_3$. We write $\mathcal{P}(a) = X/P'$. Then $a$ acts trivially on $X/P'$ and on $X/P'$, so $a$ acts trivially on $X$ (cf. Theorem 5.3.2 in [1]). Thus $X = \mathcal{P}(a)$. Any other involution $a'$ in $G$ is conjugate to $a$ by an element in $P$. Hence $\mathcal{P}(a') = \mathcal{P}(a) = X$. But $S$ is generated by its involutions, so $X \subseteq \mathcal{P}(S)$. This means that $G = X \times S$, where $|X'| = 3$. However, this is impossible by Lemma 6.

We are thus left with the case when $G$ is a finite 3-group. Recall that we may write $G' = \langle (a, b), (c, d) \rangle$ and $G = \langle a, b, c, d \rangle$, so that $|G/\Phi(G)| \leq 3^4$. We assume first that $|G/\Phi(G)| = 3^4$. (The following argument is due to Zhang.) Then $\langle a, b, c \rangle \neq G$, so $\langle a, b, c \rangle' = \langle (a, b) \rangle$. Similarly, $\langle a, b, d \rangle' = \langle (a, b), (c, d) \rangle$, and $\langle b, c, d \rangle' = \langle (c, d) \rangle$. Hence $(a, c) \in \langle (a, b) \rangle \cap \langle (c, d) \rangle = 1$. Similarly, $1 = (a, d) = (b, c) = (b, d)$. But now $|\langle a, b, c, d \rangle'| = 9$, so $G = \langle a, b, c, d \rangle$ and $|G/\Phi(G)| \leq 3^3$, a contradiction.

Suppose next that $|G/\Phi(G)| = 3^2$, and write $G = \langle a, b \rangle$. Since $|G/\mathcal{F}| \leq 3$ we may assume that $b \in \mathcal{F}$. Then $G'$ is generated by $c := (a, b)$ and $\gamma_3(G)$. And $\gamma_3(G)$ is generated by $d := (a, c)$ since $(a, c) = 1$. It is now easy (but lengthy) to verify that $[ [ab - ba, ac - ca], b ] \neq 0$, so we get a contradiction in this case.

It remains to deal with the case $|G/\Phi(G)| = 3^3$. If $x, y \in G$, then $\langle x, y \rangle \neq G$, so there exists a maximal subgroup $M$ of $G$ containing $\langle x, y \rangle$. Thus $|M'| \leq 3$, so $(x, y) \in M' \subseteq \mathcal{F}G$; in particular, $G$ is nilpotent of class 2. We write $G = \langle a, b, c \rangle$. Then $G' = \langle (a, b), (a, c), (b, c) \rangle$. Since $|G'| \leq 9$, we may assume that $G' = \langle (a, b), (a, c) \rangle$, and we can write $(b, c) = (a, b)^i(a, c)^j$ with integers $i, j$. Then we set $\bar{b} := a^{-1}b$, $\bar{c} := a'c$, so that $G = \langle a, \bar{b}, \bar{c} \rangle$, $(a, b) = (a, c) = (a, c)$, and

$$(\bar{b}, \bar{c}) = (a^{-1}b, a'c) = (a, c)^{-i}(a, b)^i(b, c) = 1.$$  

Thus we can replace $b$ by $\bar{b}$ and $c$ by $\bar{c}$ to obtain $(b, c) = 1$. An easy (but lengthy) calculation shows that $[ [ab - ba, ac - ca], b ] \neq 0$ again. This means that we have reached our final contradiction, and hence Lemma 7 is proved.

Theorem 1 now follows from Lemmas 3 and 7.

It follows from Lemma 3 that if $F$ is a field of characteristic 3 and $G$ is a group with $|G'| \leq 3$ then the group of units $\mathbb{U}(FG)$ of $FG$ is centrally metabelian. This may be the best possible result in this direction. We give the following example in order to substantiate this fact.
Example 8. Let $F$ be a field of characteristic 3 and $S_3$ be the symmetric group on 3 symbols, then the group of units $\mathcal{U}(FS_3)$ of $FS_3$ is centrally metabelian but not metabelian.

Proof. Let $\mathcal{A} = FS_3$ and $\mathcal{U} = \mathcal{U}(\mathcal{A})$, then $\mathcal{U}$ is centrally metabelian by Lemma 3.

We now show that $\mathcal{U}$ is not metabelian. Let $S_3 = \langle g, h : g^3 = h^2 = 1, hgh^{-1} = g^2 \rangle$ and $\mathcal{F} = \Delta(FS_3')FS_3$. The elements $e = -1 - h$ and $f = 1 - e$ are two orthogonal idempotents in $\mathcal{A}$ such that $e + f = 1$, and hence the two-sided Peirce decomposition of $\mathcal{A}$ can be written as

$$\mathcal{A} = e\mathcal{A}e \oplus e\mathcal{F}f \oplus f\mathcal{A}e \oplus f\mathcal{F}f.$$  

Also, for some suitable $a, b \in \mathcal{A}$ we can write

$$e\mathcal{A}e = Fe + Fab, \quad e\mathcal{F}f = Fa, \quad f\mathcal{A}e = Fb, \quad f\mathcal{F}f = Ff + Fba$$

and

$$\mathcal{F} = Fab + Fa + Fb + Fba.$$  

Now, $\mathcal{F}^2$ is central in $\mathcal{A}$. For any $\alpha, \beta \in F$, the following relations hold in $\mathcal{A}$ modulo $\mathcal{F}^2$

$$((e - f), (1 + \alpha a + \beta b))$$

$$= (e - f)(1 + \alpha a + \beta b)(e - f)^{-1}(1 + \alpha a + \beta b)^{-1}$$

$$= (e - f)(1 + \alpha a + \beta b)(e - f)(1 - \alpha a - \beta b)$$

$$= (e + \alpha a - f - \beta b)(e - \alpha a - f + \beta b)$$

$$= e - \alpha a + \alpha a + f - \beta b - \beta b$$

$$= 1 + \alpha a + \beta b.$$  

This implies $\mathcal{U}' + \mathcal{F}^2 \supseteq 1 + Fa + Fb + \mathcal{F}^2 = 1 + \mathcal{F} \supseteq \mathcal{U}' + \mathcal{F}^2$. In particular, $\mathcal{U}' + \mathcal{F}^2 = 1 + \mathcal{F}$. If $\mathcal{U}$ is metabelian, then $[\mathcal{U}', \mathcal{U}'] = 0$ and since $\mathcal{F}^2$ is central in $\mathcal{A}$, this implies that $[\mathcal{U}' + \mathcal{F}^2, \mathcal{U}' + \mathcal{F}^2] = 0$. This further implies that $[\mathcal{F}, \mathcal{F}] = [1 + \mathcal{F}, 1 + \mathcal{F}] = 0$. Hence $\mathcal{F}$ must be commutative. But $(g - 1), (g - 1)h \in \mathcal{F}$ are such that

$$[(g - 1), (g - 1)h]$$

$$= (g - 1)^2 h - (g - 1)h(g - 1) = (g - 1)^2 h - (g - 1)(g^2 - 1)h$$

$$= (g - 1)^2 h - (g - 1)^2(g + 1)h = (g - 1)^2(1 - g - 1)h$$

$$= -(g - 1)^2 gh \neq 0.$$  

Hence $\mathcal{U}$ is not metabelian.
We now concentrate on Theorem 2. An ideal \( U \) of the associated Lie ring \( A(R) \) of the ring \( R \) is called a Lie ideal of \( R \). Thus a Lie ideal \( U \) of \( R \) is an additive subgroup of \( R \) such that \([U, R] \subseteq U\). Let \( R \) be a Lie solvable ring of length \( n \geq 3 \), \( c = \frac{2}{9}(19.10^{n-3} - 1) \), \( J \) be the ideal of \( R \) generated by all elements of the type \([[x, y], [z, u], v]] \); \( x, y, z, u, v \in R \), and \( I \) be the derived ideal of \( R \) generated by all elements of the type \([\alpha, \beta]; \alpha, \beta \in R\). It is proved in [4] that if \( 3 \) is invertible in \( R \) then \( J \) is nilpotent of index at most \( c \) and when \( 2 \) and \( 3 \) both are invertible in \( R \) then \( I \) is a nil ideal in \( R \). Theorem 2 proves this result without the restriction of invertibility of \( 3 \). However, the invertibility of \( 2 \) in the context cannot be dropped. We first prove the following:

**Lemma 9.** Let \( U \) be a Lie ideal of \( R \), then \((\gamma_3(U))^2 \subseteq \delta^{(2)}(U)R\).

**Proof.** The following identity holds true for any elements \( x_1, x_2, x_3, y_1, y_2, y_3 \in R:\)

\[
2[x_1, x_2, x_3][y_1, y_2, y_3] = [[x_1, x_2], [y_3, y_2, y_1, x_3]] + [[x_1, x_2], [y_3, y_2, y_1, x_3]]
+ [[x_1, x_2], [y_3, y_2, y_1, x_3]] - [[y_1, y_2], [x_3, x_1, x_2, y_3]]
- [[y_1, y_2], [x_3, x_1, x_2, y_3]] + [[y_3, x_3], [x_2, x_1, y_1, y_2]] + [[y_3, x_3], [x_2, x_1, y_1, y_2]]
+ [[y_1, x_2], [x_3, y_2, x_3, y_1]] - [[y_3, x_3], [x_2, y_2, x_1, y_1]]
+ [[y_3, x_3], [x_2, y_2, x_1, y_1]] - [[y_3, x_3], [x_2, y_2, x_1, y_1]] + [[y_1, x_2], [x_3, y_2, x_3, y_1]] + [[y_1, x_2], [x_3, y_2, x_3, y_1]]
+ [[y_3, x_3, y_2, x_2, x_1, y_1]] - [[y_3, x_3, x_1, y_2, x_2, y_1]]
- [[y_1, x_2], [y_3, x_3, y_2, x_2, y_1]] + [[y_1, x_2], [y_3, x_3, x_1, y_2, x_2, y_1]]
+ x_3[[x_1, x_2], [y_3, y_1, y_2]] + [[y_1, y_2], [x_3, x_1, x_2]]y_3
- y_3[[y_1, y_2], [x_3, x_2, x_1]] - [[y_3, x_3], [x_2, y_1, y_2]]x_1
+ x_1[[y_3, x_3], [x_2, y_2, y_1]].
\]

Now by letting the elements \( x_1, x_2, x_3, y_1, y_2, y_3 \in U \) in the above identity and using the fact that \( RUR = UR = RU \) for any Lie ideal \( U \) of \( R \), we get that \( 2(\gamma_3(U))^2 \subseteq \delta^{(2)}(U)R \). Also, it follows from the Proof of Lemma 1.7 of [4] that \( 3(\gamma_3(U))^2 \subseteq \delta^{(2)}(U)R \). On combining the two, we get Lemma 9.
The proof of Theorem 2 follows on the same lines as those in [4] by replacing Lemma 1.7 in [4] by the above Lemma 9.

Finally, since for a Lie solvable ring R, the ideal J is nilpotent (of index at most c), therefore \( \mathcal{Z}(R/J) = (\mathcal{Z}(R) + J)/J \), where \( \mathcal{Z}(A) \) denotes the group of units for a ring A. In fact, for any positive integer \( n \), we have \( \delta^{(n)}(\mathcal{Z}(R/J)) = (\delta^{(n)}(\mathcal{Z}(R)) + J)/J \). Hence \( \mathcal{Z}(R) \) is solvable if and only if \( \mathcal{Z}(R/J) \) is solvable. But \( R/J \) is a Lie centrally metabelian ring. Obviously if one knows the solvability length of the group of units of a Lie centrally metabelian ring one then knows a bound for the solvability length of the group of units of an arbitrary Lie solvable ring. Thus their study becomes important.

REFERENCES