A note on generalized \((\alpha, \beta)\)-derivation

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Abstract

The purpose of this note is to prove that every multiplicative generalized \((\alpha, \beta)\)-derivation on a ring \(R\) is additive if there exists an idempotent \(e' (e' \neq 0,1)\) in \(R\) (need not have an identity) satisfying certain conditions. Moreover, we will prove that any multiplicative generalized \((\alpha, \beta)\)-derivation on the algebra \(M_n(\mathbb{C})\) of all \(n \times n\) complex matrices can be decomposed into a sum of a generalized \((\alpha, \beta)\)-inner derivation and a generalized \((\alpha, \beta)\)-derivation on \(M_n(\mathbb{C})\) given by an additive generalized derivation on \(\mathbb{C}\).

Keywords: Ring, \((\alpha, \beta)\)-derivation, Generalized \((\alpha, \beta)\)-derivation, Peirce decomposition.

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1 Introduction

Let \(R\) be an associative ring, \(\alpha\) and \(\beta\) be the automorphisms on \(R\). By a derivation, we mean an additive mapping \(D : R \rightarrow R\) such that \(D(xy) = D(x)y + xD(y)\) for all \(x, y \in R\) and if we remove the additivity condition, then \(D\) is called multiplicative derivation on \(R\). An additive mapping \(F : R \rightarrow R\) associated with a derivation \(D\) on \(R\) such that \(F(xy) = F(x)y + xD(y)\) for all \(x, y \in R\), is said to be generalized derivation.

An additive mapping \(d\) on \(R\) is called \((\alpha, \beta)\)-derivation if \(d(xy) = d(x)\alpha(y) + \beta(x)d(y)\) holds for all \(x, y \in R\). A map \(d : R \rightarrow R\) is called \((\alpha, \beta)\)-inner derivation if there exists \(b \in R\) such that \(d(x) = \beta(x)b - \beta(a(x))\) for all \(x \in R\).

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A mapping \( g : R \to R \) is said to be multiplicative generalized \((\alpha, \beta)\)-derivation if there exists an \((\alpha, \beta)\)-derivation \( d \) on \( R \) such that \( g(xy) = g(x)\alpha(y) + \beta(x)d(y) \) for all \( x, y \in R \). Moreover, if \( g \) is additive, then we call \( g \) a generalized \((\alpha, \beta)\)-derivation on \( R \). A map on a ring \( R \) defined as \( x \mapsto a\alpha(x) + \beta(x)b \), where \( a, b \) are fixed elements of \( R \), is called generalized \((\alpha, \beta)\)-inner derivation, is an example of generalized \((\alpha, \beta)\)-derivation.

Additive mappings are closely connected with the structure of rings. The first pioneer result was proved by Posner [12]. He established relationship between commutativity of ring and derivation on the same ring. Several results in literature studied the structure of prime rings, semiprime rings and some other rings admitting derivations, generalized derivations, automorphisms and many other maps satisfying certain algebraic conditions on appropriate subsets of the rings.

For these kind of results, we refer the reader to (Bell et al. [2]; Daif and Tammam [4]; Rehman [13]; Marubayashi et al. [11]; Filippis and Dhara [5]; Filippis and Dhara [6]; Scudo [14]; Garg and Sharma [7]; Ali and Khan [1], where further references can be found).

In this article, we will study the additivity of some special type of multiplicative mappings. In general, multiplicative mappings are not additive. It is natural to consider the question when a multiplicative mapping is additive. Firstly, this question was considered by Martindale [10] in 1969. He gave answer of this question for a multiplicative isomorphism. Daif [3] gave answer of the same question for multiplicative derivation in 1991. Daif [4] and Hou, C. et al. [8] answered to the above question for multiplicative generalized derivation and multiplicative \((\alpha, \beta)\)-derivation, respectively.

In the similar fashion, we will investigate the additivity of multiplicative generalized \((\alpha, \beta)\)-derivation. More precisely, we will prove that every multiplicative generalized \((\alpha, \beta)\)-derivation on a ring \( R \) under the existence of an idempotent satisfying some conditions which are similar to Daif’s conditions in [4], is additive.

\section{Additivity of multiplicative generalized \((\alpha, \beta)\)-derivations on rings}

In this section, we will prove the following main result.

**Theorem 2.1.** Let \( R \) be a ring, \( \alpha \) and \( \beta \) be the automorphisms on \( R \). Suppose that there exists an idempotent \( e' \) \((e' \neq 0, 1)\) satisfying the following conditions:

- \((T1)\) \( xRe = 0 \) implies \( x = 0 \) (and hence \( xR = 0 \) implies \( x = 0 \));
- \((T2)\) \( exeR(1-e) = 0 \) implies \( exe = 0 \);
- \((T3)\) \((1-e)xeR(1-e) = 0 \) implies \((1-e)xe = 0\);

where \( e = \alpha(e') \), then every multiplicative generalized \((\alpha, \beta)\)-derivation on \( R \) is
Corollary 2.2 ([4], Theorem 2.1). Let $R$ be a ring containing an idempotent $e$ ($e \neq 0, 1$) satisfying the following conditions:

(N1) $xeR = 0$ implies $x = 0$ (and hence $xR = 0$ implies $x = 0$);
(N2) $exeR(1 - e) = 0$ implies $exe = 0$;
(N3) $(1 - e)xR(1 - e) = 0$ implies $(1 - e)xe = 0$;

then every multiplicative generalized derivation on $R$ is additive.

Proof. Replacing $\alpha$ and $\beta$ by identity map on $R$ in theorem 2.1, we get the required result. \hfill $\square$

Let $g : R \to R$ be a multiplicative generalized $(\alpha, \beta)$-derivation associated with $(\alpha, \beta)$-derivation $d$ on $R$. Suppose that $\alpha(e') = e$ and $\beta(e') = \bar{e}$. As in [9], the two-sided Peirce decomposition of $R$ relative to the idempotent $e'$ takes the form $R = R'_{11} \oplus R'_{12} \oplus R'_{21} \oplus R'_{22}$, where $R'_{11} = e'Re'$, $R'_{12} = e'R(1 - e')$, $R'_{21} = (1 - e')Re'$, $R'_{22} = (1 - e')R(1 - e')$. An element of the subring $R'_{mn}$, $m,n=1,2$, will be denoted by $x'_{mn}$. Also, note that $R = \alpha(R'_{11}) \oplus \alpha(R'_{12}) \oplus \alpha(R'_{21}) \oplus \alpha(R'_{22})$. Relative to the idempotents $e$ and $\bar{e}$, we have the generalized two-sided Peirce decomposition of $R$, $R = R_{11} \oplus R_{12} \oplus R_{21} \oplus R_{22}$, where $R_{11} = eRe$, $R_{12} = eR(1 - e)$, $R_{21} = (1 - e)Re$, $R_{22} = (1 - e)R(1 - e)$ given by Hou, C. et al. [8]. An element of the subring $R_{mn}$ will be denoted by $x_{mn}$.

From the definition of $g$, we see that $g(0) = 0$ and also, $d(0) = 0$. Now, $d(e') = d(e')e + \bar{e}d(e')$. Let $d(e') = a_{11} + a_{12} + a_{21} + a_{22}$. Using the value of $d(e')$, we get $a_{11} = a_{22} = 0$. Consequently, we have $d(e') = a_{12} + a_{21}$. For simplifications, let $f$ be $(\alpha, \beta)$-inner derivation given by the element $a_{12} - a_{21}$, that is $f(x) = \beta(x)(a_{12} - a_{21}) - (a_{12} - a_{21})\alpha(x)$. Clearly $f(e') = a_{12} + a_{21}$. In the same manner, $g(e') = g(e')e + \bar{e}d(e')$. Let $g(e') = b_{11} + b_{12} + b_{21} + b_{22}$. Using the values of $g(e')$ and $d(e')$, we have $b_{11} + b_{12} + b_{21} + b_{22} = b_{11} + a_{12} + b_{21}$. Uniqueness of representation of an element in a direct sum implies that $b_{22} = 0$ and $b_{12} = a_{12}$. Consequently, we have $g(e') = b_{11} + a_{12} + b_{21}$. Again, for the simplifications, let $F(x) = (b_{11} + b_{21})\alpha(x) + \beta(x)(a_{12} - a_{21})$ be the generalized $(\alpha, \beta)$-inner derivation given by the elements $b_{11} + b_{21}$ and $a_{12} - a_{21}$. Note that $F(e') = b_{11} + a_{12} + b_{21}$.

Suppose that $G = g - F$. We note that $G$ is a multiplicative generalized $(\alpha, \beta)$-derivation associated with $(\alpha, \beta)$-derivation $D = d - f$. Clearly, $G(e') = D(e') = 0$. We see that $G$ is additive if and only if $g$ is additive. Hence, it is sufficient to show that $G$ is additive.

Lemma 2.3. ([8], Lemma 1) $D(R'_{mn}) \subseteq R_{mn}$, $m,n=1,2$.
Lemma 2.4. \(G(R_{1n}) \subseteq R_{1n}, n = 1, 2; G(R'_{21}) \subseteq R_{11} + R_{21}; G(R'_{11} + R'_{21}) \subseteq R_{11} + R_{21}\) and \(G(R'_{22}) \subseteq R_{22} + R_{12}\). Moreover, \(G\) is additive on \(R_{1n}\) and \(G(x'_{11} + x'_{12}) = G(x'_{11}) + G(x'_{12})\) for all \(x'_{11} \in R_{11}\) and \(x'_{12} \in R'_{11}\).

Proof. Let \(x'_{1n} \in R'_{1n}, n = 1, 2\). Using Lemma 2.3 and \(G(e') = 0\), we have \(G(x'_{1n}) = G(x'_{1n}) = G(e') \alpha(x'_{11}) + eD(x'_{1n}) = eD(x'_{1n}) = D(x'_{1n}) \in R_{1n}\). Thus, \(G(x'_{1n}) = D(x'_{1n})\), clearly \(G\) is additive on \(R_{1n}\). Again, using \(G(x'_{1n}) = D(x'_{1n})\), we find \(G(x'_{11} + x'_{12}) = G(e'(x'_{11} + x'_{12})) = eD(x'_{11} + x'_{12}) = D(x'_{11} + x'_{12}) = D(x'_{11}) + D(x'_{12}) = G(x'_{11}) + G(x'_{12})\). Thus, \(G(x'_{11} + x'_{12}) = G(x'_{11}) + G(x'_{12})\). Hence, suppose that \(G(x'_{21}) = b_{11} + b_{21} + b_{22}\). Since \(G(x'_{21}) = G(x'_{21})\), it follows that \(G(x'_{21}) = (b_{11} + b_{21} + b_{22}) e = (b_{11} + b_{21}) \in R_{11} + R_{21}\). Hence, \(G(R'_{21}) \subseteq R_{11} + R_{21}\). If \(x'_{11} \in R'_{11}\) and \(x'_{12} \in R'_{21}\), then \(G(x'_{11} + x'_{12}) = G((x'_{11} + x'_{12}) e) = (G(x'_{11} + x'_{12})) e = R_{11} + R_{21}\). So, \(G(R'_{11} + R'_{21}) \subseteq R_{11} + R_{21}\). Finally for \(x'_{22} \in R'_{22}\), let \(G(x'_{22}) = a_{11} + a_{12} + a_{21} + a_{22}\), then using \(0 = G(x'_{22}) = (a_{11} + a_{12} + a_{21} + a_{22}) e = (a_{11} + a_{21})\), so \(G(x'_{22}) = a_{22} + a_{12} \in R_{22} + R_{12}\). Hence, proof of the lemma is completed.

Lemma 2.5. For any elements \(x'_{11} \in R'_{11}, x'_{21} \in R'_{21}\) and \(x'_{12} \in R'_{12}\), we have \(G(x'_{21} + x'_{11}z'_{12}) = G(x'_{21}) + G(x'_{11}z'_{12})\).

Proof. For \(t_1 \in \mathbb{C}R\), let \(t_1 = \alpha(s_1) = e\alpha(s_1) = \alpha(e's_1)\), so \(s_1 = e's_1\). Lemma 2.4 gives \((G(x'_{11}z'_{12})) e = 0\). Hence, using \(D(e') = 0\), we have

\[
\{G(x'_{21}) + G(x'_{11}z'_{12})\} t_1 = \{G(x'_{21}) + G(x'_{11}z'_{12})\} \alpha(e's_1) = G(x'_{21}) \alpha(e's_1) + G(x'_{11}z'_{12}) \alpha(e') \alpha(s_1) = G(x'_{21}) \alpha(s_1) = \{G(x'_{21}) + G(x'_{11}z'_{12})\} t_1 = G(x'_{21} + x'_{11}z'_{12}) \alpha(e') t_1 = \{G(x'_{21} + x'_{11}z'_{12})\} t_1
\]

Thus, we get

\[
\{G(x'_{21} + x'_{11}z'_{12}) - G(x'_{21}) - G(x'_{11}z'_{12})\} t_1 = 0
\]

Now, for \(t_2 \in (1 - e)R\), let \(t_2 = \alpha(s_2) = (1 - e)\alpha(s_2) = \alpha(s_2) = \alpha(e's_2) = \alpha((1 - e')s_2)\), so \(s_2 = (1 - e's_2)\). Using Lemma 2.4, \(G(x'_{21}) \in R_{11} + R_{21}\). Consequently, \(G(x'_{21}) t_2 = G(x'_{21})(1 - e)\alpha(s_2) = 0\). Hence, using \(D(e') = 0\), we have

\[
\{G(x'_{21}) + G(x'_{11}z'_{12})\} t_2 = \{G(x'_{11}z'_{12})\} t_2 = G(x'_{11}z'_{12}) \alpha((1 - e')s_2) = G(x'_{11}z'_{12}(1 - e')) \alpha(s_2) = G((x'_{21} + x'_{11}z'_{12})(1 - e')) \alpha(s_2) = G(x'_{21} + x'_{11}z'_{12}) \alpha((1 - e')s_2) = (G(x'_{21} + x'_{11}z'_{12})) t_2
\]

So, we obtain

\[
\{G(x'_{21} + x'_{11}z'_{12}) - G(x'_{21}) - G(x'_{11}z'_{12})\} t_2 = 0
\]
Since $t_1$, $t_2$ are arbitrary, from (2) and (4), we get

$$\{G(x'_{21} + x'_{11}z'_{12}) - G(x'_{21}) - G(x'_{11}z'_{12})\} t = 0 \tag{5}$$

for all $t \in R$. From condition $(T1)$, it follows that

$$G(x'_{21} + x'_{11}z'_{12}) = G(x'_{21}) + G(x'_{11}z'_{12}) \tag{6}$$

Thereby the proof of the lemma is completed.

Lemma 2.6. For any elements $x'_{11} \in R'_{11}$ and $x'_{21} \in R'_{21}$, we have $G(x'_{11} + x'_{21}) = G(x'_{11}) + G(x'_{21})$.

Proof. Suppose that $u'_{in} \in R'_{11}$ and $z'_{12} \in R'_{12}$, $n = 1, 2$. Clearly, we have

$$\{G(x'_{11} + x'_{21}) - G(x'_{11}) - G(x'_{21})\} \alpha(z'_{12}) \alpha(u'_{in}) = 0 \tag{7}$$

that means

$$\{G(x'_{11} + x'_{21}) - G(x'_{11}) - G(x'_{21})\} \alpha(z'_{12}) \alpha(R'_{11}) = (0) \tag{8}$$

Now, for any $u'_{2n} \in R'_{2n}$ and $z'_{12} \in R'_{12}$, $n = 1, 2$, using lemma 2.3 and 2.5, we obtain

$$G(x'_{11} + x'_{21}) \alpha(z'_{12}) \alpha(u'_{2n}) = G((x'_{11} + x'_{21})z'_{12}u'_{2n}) - \beta(x'_{11} + x'_{21})D(z'_{12}u'_{2n})$$

$$= G((x'_{11}z'_{12} + x'_{21}z'_{12}u'_{2n})) - \beta(x'_{11} + x'_{21})D(z'_{12}u'_{2n})$$

$$= G(x'_{11}z'_{12} + x'_{21}z'_{12}u'_{2n} + z'_{12}u'_{2n}) + \beta(x'_{11}z'_{12} + x'_{21}D(u'_{2n} + z'_{12}u'_{2n}) - \beta(x'_{11} + x'_{21})D(z'_{12}u'_{2n})$$

$$= G(x'_{11}z'_{12} + x'_{21}) \alpha(u'_{2n} + z'_{12}u'_{2n}) + \beta(x'_{11}z'_{12}D(u'_{2n} - \beta(x'_{11})D(z'_{12}u'_{2n})$$

$$= G(x'_{11}z'_{12} + x'_{21}) \alpha(u'_{2n} + z'_{12}u'_{2n}) - \beta(x'_{11})D(z'_{12}u'_{2n})$$

$$= \{G(x'_{11}z'_{12} + x'_{21}) \alpha(u'_{2n} + z'_{12}u'_{2n}) - \beta(x'_{11})D(z'_{12}u'_{2n})$$

$$= \{G(x'_{11}) + G(x'_{21})\} \alpha(z'_{12}) \alpha(u'_{2n})$$

Thus, we have

$$\{G(x'_{11} + x'_{21}) - G(x'_{11}) - G(x'_{21})\} \alpha(R'_{2n}) = (0) \tag{10}$$

Combining (8) and (10), we get

$$\{G(x'_{11} + x'_{21}) - G(x'_{11}) - G(x'_{21})\} \alpha(z'_{12}) R = (0) \tag{11}$$

Using condition $(T1)$, we have

$$\{G(x'_{11} + x'_{21}) - G(x'_{11}) - G(x'_{21})\} \alpha(R'_{12}) = (0) \tag{12}$$
Using conditions (T2) and (T3), we obtain
\[ G(x'_{11} + x'_{21}) = G(x'_{11}) + G(x'_{21}) \] (13)

This completes the proof of the lemma. \( \square \)

**Lemma 2.7.** For any elements \( x'_{21}, y'_{21} \in R'_{21} \) and \( z'_{12} \in R'_{12} \), we have \( G(y'_{21} + x'_{21}z'_{12}) = G(y'_{21}) + G(x'_{21}z'_{12}) \).

**Proof.** For \( t_1 \in eR \), let \( t_1 = \alpha(s_1) = e\alpha(s_1) = \alpha(e's_1) \), so \( s_1 = e's_1 \). Using lemma 2.4, we see that \( G(x'_{21}z'_{12}) \in R_{12} + R_{22} \). This gives \( (G(x'_{21}z'_{12})t_1 = (G(x'_{21}z'_{12}))et_1 = 0 \). Hence, using \( D(e') = 0 \), we have
\[
\{G(y'_{21}) + G(x'_{21}z'_{12})\}t_1 = (G(y'_{21})t_1 = G(y'_{21})\alpha(e's_1) = G(y'_{21}e')\alpha(s_1) = G((y'_{21} + x'_{21}z'_{12})e')\alpha(s_1) = G(x'_{21} + x'_{21}z'_{12})\alpha(e')\alpha(s_1) = G(x'_{21} + x'_{21}z'_{12})t_1
\] (14)

Thus, we get
\[
\{G(y'_{21}) + x'_{21}z'_{12} - G(y'_{21}) - G(x'_{21}z'_{12})\}t_1 = 0 \] (15)

Now, for \( t_2 \in (1-e)R \), let \( t_2 = (1-e)\alpha(s_2) = \alpha(s_2) - \alpha(e's_2) = \alpha((1-e')s_2) \), so \( s_2 = (1-e')s_2 \). Using lemma 2.4, \( G(y'_{21}) \in R_{11} + R_{21} \). Consequently, \( G(y'_{21})t_2 = G(y'_{21})(1-e)\alpha(s_2) = 0 \). Hence, using \( D(e') = 0 \), we have
\[
\{G(y'_{21}) + G(x'_{21}z'_{12})\}t_2 = G(x'_{21}z'_{12})\alpha((1-e')s_2) = G(x'_{21}z'_{12})(1-e')\alpha(s_2) = G(y'_{21} + x'_{21}z'_{12})\alpha(1-e')s_2 = (G(y'_{21} + x'_{21}z'_{12}))t_2
\] (16)

So, we obtain
\[
\{G(y'_{21}) + x'_{21}z'_{12} - G(y'_{21}) - G(x'_{21}z'_{12})\}t_2 = 0 \] (17)

Since \( t_1, t_2 \) are arbitrary, combining (15) and (17), we get
\[
\{G(y'_{21}) + x'_{21}z'_{12} - G(y'_{21}) - G(x'_{21}z'_{12})\}t = 0 \] (18)

for all \( t \in R \). Consequently, condition (T1) forces to write
\[
G(y'_{21}) + x'_{21}z'_{12} = G(y'_{21}) + G(x'_{21}z'_{12}) \] (19)

Thereby the proof of the lemma is completed. \( \square \)

**Lemma 2.8.** \( G \) is additive on \( R'_{21} \).
Proof. For any elements $x'_{21}, y'_{21} \in R'_{21}$, $z'_{12} \in R'_{12}$ and $u'_{1n} \in R'_{1n}$, $n = 1, 2$, clearly, we have

$$\{G(x'_{21} + y'_{21}) - G(x'_{21}) - G(y'_{21})\} \alpha(z'_{12}) \alpha(u'_{1n}) = 0 \quad (20)$$

which means

$$\{G(x'_{21} + y'_{21}) - G(x'_{21}) - G(y'_{21})\} \alpha(z'_{12}) \alpha(R'_{1n}) = (0) \quad (21)$$

Now, for any elements $x'_{21}, y'_{21} \in R'_{21}$, $z'_{12} \in R'_{12}$ and $u'_{2n} \in R'_{2n}$, $n = 1, 2$, using lemma 2.3 and 2.7, we obtain

$$\begin{align*}
G(x'_{21} + y'_{21}) \alpha(z'_{12}) \alpha(u'_{2n}) &= G((x'_{21} + y'_{21}) \ z'_{12} \ u'_{2n}) - \beta(x'_{21} + y'_{21}) D(z'_{12} u'_{2n}) \\
&= G((x'_{21} z'_{12} + y'_{21})(u'_{2n} + z'_{12} u'_{2n})) - \beta(x'_{21} + y'_{21}) D(z'_{12} u'_{2n}) \\
&= G(x'_{21} z'_{12} + y'_{21}) \alpha(u'_{2n} + z'_{12} u'_{2n}) + \beta(x'_{21} z'_{12} + y'_{21}) D(u'_{2n} + z'_{12} u'_{2n}) \\
&\quad - \beta(x'_{21} + y'_{21}) D(z'_{12} u'_{2n}) \\
&= G(x'_{21} z'_{12} + y'_{21}) \alpha(u'_{2n} + z'_{12} u'_{2n}) - \beta(x'_{21}) D(z'_{12}) \alpha(u'_{2n}) \\
&\quad + G(y'_{21}) \alpha(z'_{12}) \alpha(u'_{2n}) - \beta(x'_{21}) D(z'_{12}) \alpha(u'_{2n}) \\
&= \{G(x'_{21}) + G(y'_{21})\} \alpha(z'_{12}) \alpha(u'_{2n})
\end{align*} \quad (22)$$

Thus, we have

$$\{G(x'_{21} + y'_{21}) - G(x'_{21}) - G(y'_{21})\} \alpha(z'_{12}) \alpha(R'_{2n}) = (0) \quad (23)$$

Combining (21) and (23), we get

$$\{G(x'_{21} + y'_{21}) - G(x'_{21}) - G(y'_{21})\} \alpha(z'_{12}) R = (0) \quad (24)$$

Using condition (T1), we have

$$\{G(x'_{21} + y'_{21}) - G(x'_{21}) - G(y'_{21})\} \alpha(R'_{12}) = (0) \quad (25)$$

Using conditions (T2) and (T3), we obtain

$$G(x'_{21} + y'_{21}) = G(x'_{21}) + G(y'_{21}) \quad (26)$$

Hence, $G$ is additive on $R'_{21}$. \hfill $\Box$

**Lemma 2.9.** $G$ is additive on $Rc' = R'_{11} \oplus R'_{21}$

**Proof.** For any elements $x'_{11}, y'_{11} \in R'_{11}$ and $x'_{21}, y'_{21} \in R'_{21}$. Application of lemma 2.4, 2.6 and 2.8 gives $G((x'_{11} + x'_{21}) + (y'_{11} + y'_{21})) = G(x'_{11} + y'_{11} + (x'_{21} + y'_{21})) = G(x'_{21} + y'_{21}) + G(x'_{21} + y'_{21}) = G(x'_{11} + G(y'_{21}) + G(y'_{21}) = G(x'_{11} + x'_{21}) + G(y'_{11} + y'_{21})$. Thus, $G$ is additive on $Rc'$. \hfill $\Box$
The proof of theorem 2.1: For any \(x, y \in R\) and \(t \in Re\), let \(t = \alpha(s) = \alpha(s)e = \alpha(se')\), so \(s = se'\). Consequently, \(xs, ys \in Re\). Hence, using lemma 2.9, we have \((G(x) + G(y))t = (G(x) + G(y))\alpha(s) = (G(xs) + G(ys)) = (\beta(x) + \beta(y))D(s) = G((x + y)s) - \beta(x + y)D(s) = G(x + y)s\). Consequently, \((G(x) + G(y) - G(x + y))t = 0\). Thus, using condition \((T1)\), \(G\) is additive.

3 Multiquadric generalized \((\alpha, \beta)\)-derivations on \(M_n(\mathbb{C})\)

Let \(A\) be an associative algebra over \(\mathbb{C}\), \(\alpha\) and \(\beta\) be algebraic automorphisms on \(A\). Recall that \(G : A \rightarrow A\) is said to be multiplicative generalized \((\alpha, \beta)\)-derivation on \(A\) if there exists an \((\alpha, \beta)\)-derivation \(D\) on \(A\) such that \(G(xy) = G(x)\alpha(y) + \beta(x)D(y)\) for all \(x, y \in A\). It is well known that each algebraic automorphism on \(M_n(\mathbb{C})\) is inner. Thus, there exists invertible matrices \(T_0\) and \(S_0\) in \(M_n(\mathbb{C})\) such that \(\alpha(A) = T_0AT_0^{-1}\) and \(\beta(A) = S_0AS_0^{-1}\) for each \(A \in M_n(\mathbb{C})\), respectively. In 2010, Hou, C. et al. [8] proved that every multiplicative \((\alpha, \beta)\)-derivation \(D\) on \(M_n(\mathbb{C})\) can be decomposed into sum of an \((\alpha, \beta)\)-inner derivation and a \((\alpha, \beta)\)-derivation on \(M_n(\mathbb{C})\). We will extend this result on multiplicative generalized \((\alpha, \beta)\)-derivation on \(M_n(\mathbb{C})\).

Remark: Since \(M_n(\mathbb{C})\) is a prime ring, so theorem 2.1 implies that every multiplicative generalized \((\alpha, \beta)\)-derivations on \(M_n(\mathbb{C})\) is additive.

Theorem 3.1. Let \(G\) be a multiplicative generalized \((\alpha, \beta)\)-derivation associated with \((\alpha, \beta)\)-derivation \(D\) on \(M_n(\mathbb{C})\). If \(G\) is linear, then \(G\) is inner.

Proof. Let \(E_{ij}\), \(i, j = 1, 2, ..., n\), be the standard matrix unit of \(M_n(\mathbb{C})\). Let \(L_0 = \sum_{j=1}^{n} G(E_{j1})\alpha(E_{1j})\) and \(M_0 = \sum_{j=1}^{n} \beta(E_{j1})D(E_{1j})\). Then, for each \(E_{kl}\), we get

\[
L_0\alpha(E_{kl}) + \beta(E_{kl})M_0 = \sum_{j=1}^{n} G(E_{j1})\alpha(E_{1j})\alpha(E_{kl}) + \sum_{j=1}^{n} \beta(E_{j1})\beta(E_{j1})D(E_{1j})
\]

\[
= G(E_{k1})\alpha(E_{1k}) + \beta(E_{k1})D(E_{1k}) = G(E_{kl})
\]

(27)

Hence, linearity of \(G\) shows that \(G\) is a generalized \((\alpha, \beta)\)-inner derivation determined by the elements \(L_0\) and \(M_0\) associated with the \((\alpha, \beta)\)-inner derivation \(D\) determined by \(M_0\).

Lemma 3.2. [8], Lemma 6] Let \(D\) be a multiplicative \((\alpha, \beta)\)-derivation on \(M_n(\mathbb{C})\). Then, there exists an additive derivation \(f : \mathbb{C} \rightarrow \mathbb{C}\) and an invertible matrix \(V_0 = S_0T_0^{-1}\) in \(M_n(\mathbb{C})\) such that \(D(I) = f(t)V_0\) holds for all \(t\) in \(\mathbb{C}\).
Theorem 3.3. Any multiplicative generalized \((\alpha, \beta)\)-derivation \(G\) on \(M_n(\mathbb{C})\) associated with \((\alpha, \beta)\)-derivation \(D\) on \(M_n(\mathbb{C})\) can be decomposed into a sum of a generalized \((\alpha, \beta)\)-inner derivation and generalized \((\alpha, \beta)\)-derivation on \(M_n(\mathbb{C})\) given by an additive generalized derivation \(f'\) on \(\mathbb{C}\).

Proof. Since \(D\) is \((\alpha, \beta)\)-derivation on \(M_n(\mathbb{C})\). So, using lemma 3.2, there exists an additive derivation \(f\) on \(\mathbb{C}\) such that \(D(tI) = f(t)V_0\). Consider a map \(f'\) on \(\mathbb{C}\) defined as \(f'(t) = t + f(t)\). Clearly \(f'\) is additive generalized derivation associated with the derivation \(f\) on \(\mathbb{C}\). Define \(F'(A) = S_0(f'(a_{ij}))A^{-1}\) and \(F(A) = S_0(f(a_{ij})A^{-1}\) for each \(A = (a_{ij}) \in M_n(\mathbb{C})\). It is shown that \(F\) is an \((\alpha, \beta)\)-derivation on \(M_n(\mathbb{C})\) in theorem 4 in [8]. So, we have

\[
F'(A)\alpha(B) + \beta(A)F(B) = S_0(f'(a_{ij}))A^{-1}BTA^{-1} + S_0A(S_0^{-1}S_0(f(b_{ij}))A^{-1}
\]

\[
= S_0\left(\sum_{k=1}^{n}(f'(a_{ik})b_{kj} + a_{ik}f(b_{kj}))\right)A^{-1}
\]

\[
= S_0\left(\sum_{k=1}^{n}f'(a_{ik}b_{kj})\right)A^{-1}
\]

\[
= S_0\left(f'\left(\sum_{k=1}^{n}a_{ik}b_{kj}\right)\right)A^{-1} = F'(AB)
\]

(28)

Also, \(F'\) is additive. Thus, \(F'\) is generalized \((\alpha, \beta)\)-derivation on \(M_n(\mathbb{C})\) associated with \((\alpha, \beta)\)-derivation \(F\) on \(M_n(\mathbb{C})\). Now, let \(\tilde{G} = G - F'\). Clearly \(\tilde{G}\) is a generalized \((\alpha, \beta)\)-derivation on \(M_n(\mathbb{C})\). Also

\[
\tilde{G}(tA) = G(tA) - F'(tA) = tG(A) + \beta(A)D(tI) - f'(A) - \beta(A)tV_0 - (f(t)V_0)
\]

(29)

Consequently, \(\tilde{G}\) is linear. So, using theorem 3.1, \(\tilde{G}\) is a generalized \((\alpha, \beta)\)-inner derivation. This completes the proof of the theorem. \(\square\)

References


