Let \( \mathbb{F}_q G \) be the group algebra of a finite group \( G \) over \( \mathbb{F}_q = GF(q) \). Using the Wedderburn decomposition of \( \mathbb{F}_{2^k} D_{2n} / J(\mathbb{F}_{2^k} D_{2n}) \), we establish the structure of the unit group of \( \mathbb{F}_{2^k} G \) when \( G \) is either \( D_{4n} \), the dihedral group of order \( 4n \) or \( Q_{4n} \), the generalized quaternion group of order \( 4n \), \( n \) odd.

1. Introduction

Let \( FG \) be the group algebra of a finite group \( G \) over a field \( F \) and \( \mathcal{U}(FG) \) be its unit group. The study of the group of units is one of the classical topics in group ring theory. Results obtained in this direction are useful for the investigation of Lie properties of group rings, isomorphism problem and other open questions in this area \[1\].

In \[2\], Bovdi gave a comprehensive survey of results concerning the group of units of a modular group algebra of characteristic \( p \). There is a long tradition on the study of the unit group of finite group algebras \[3-12\]. In general, the structure of \( \mathcal{U}(FG) \) is elusive if \( |G| = 0 \) in \( F \).

Let us introduce the background of our investigation. The structure of \( \mathcal{U}(\mathbb{F}_2 D_{2p}) \) was determined by Kaur and Khan in \[13\] for an odd prime \( p \). Recently, the authors generalized this result and computed the structure of the unit group of \( \mathbb{F}_{2^k} D_{2n} \) when \( n \) is odd. In this note, we use the Wedderburn decomposition of \( \mathbb{F}_{2^k} D_{2n} / J(\mathbb{F}_{2^k} D_{2n}) \) obtained in \[14\] to study the unit group of \( \mathbb{F}_{2^k} D_{4n} \) and \( \mathbb{F}_{2^k} Q_{4n} \) when \( n \) is odd.

In what follows, \( q = 2^k \), \( ord_l(m) \) denotes the multiplicative order of \( m \) modulo \( l \) when \( (l, m) = 1 \) and \( \varphi(n) \) denotes the Euler's phi function on a positive integer \( n \).

2. Main Results

In this section, we begin by considering the lemmas that are essential for developing the proof of main results.

**Lemma 2.1.** Let \( F \) be a perfect field, \( G \) be a finite group and \( J(FG) \) be the Jacobson radical of \( FG \). Then

\[
\mathcal{U}(FG) \cong (1 + J(FG)) \rtimes \mathcal{U}\left(\frac{FG}{J(FG)}\right)
\]

**Proof.** Observe that

\[
\begin{array}{c}
1 \longrightarrow 1 + J(FG) \overset{inc}{\longrightarrow} \mathcal{U}(FG) \overset{\psi}{\longrightarrow} \mathcal{U}\left(\frac{FG}{J(FG)}\right) \longrightarrow 1
\end{array}
\]

MSC(2010): Primary: 16U60; Secondary: 16S34, 20C05.

Keywords: Group Algebra, Unit Group, Wedderburn decomposition, Jacobson radical.

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is a short exact sequence of groups, where \( \psi(x) = x + J(FG) \; \forall \; x \in U(FG) \).

By Wedderburn-Malcev theorem [15 Thm. 6.2.1], it follows that there exists a semisimple subalgebra \( B \) of \( FG \) such that

\[
FG = B \oplus J(FG)
\]

and thus for each \( x + J(FG) \in \frac{FG}{J(FG)} \), there exists a unique \( x_B \in B \) such that

\[
x + J(FG) = x_B + J(FG)
\]

Define \( \theta : U \left( \frac{FG}{J(FG)} \right) \to U(FG) \) as

\[
\theta(x + J(FG)) = x_B \; \forall \; x + J(FG) \in U \left( \frac{FG}{J(FG)} \right)
\]

Then \( \theta \) is a group homomorphism such that \( \psi \circ \theta = id \mid U(FG/J(FG)) \) and hence

\[
U(FG) \cong (1 + J(FG)) \times U \left( \frac{FG}{J(FG)} \right)
\]

For a normal subgroup \( H \) of \( G \), the natural homomorphism \( \varepsilon_H : G \to G/H \) can be extended to an \( F \)-algebra epimorphism \( \varepsilon_H^* : FG \to F(G/H) \). The kernel of \( \varepsilon_H^* \) is denoted by \( \Delta(G,H) \) and \( \Delta(G) = \Delta(G,G) \).

**Lemma 2.2.** [16 Lemma 1.17] Let \( G \) be a locally finite p-group and \( F \) be a field of characteristic \( p \). Then \( J(FG) = \Delta(G) \).

**Lemma 2.3.** [17 Ch. 1, Prop. 6.16] Let \( f : R_1 \to R_2 \) be a surjective homomorphism of rings. Then

\[
f(J(R_1)) \subseteq J(R_2)
\]

with equality if \( \ker f \subseteq J(R_1) \).

**Lemma 2.4.** [18 Theorem 7.2.7] Let \( H \) be a normal subgroup of \( G \) with \( |G : H| = n < \infty \). Then \( J(FG)^n \subseteq J(FH)FG \subseteq J(FG) \). If in addition \( n \neq 0 \) in \( F \), then \( J(FG) = J(FH)FG \).

**Lemma 2.5.** Let \( N = 2^t n \) such that \( 2 \nmid n \). Then

\[
\mathbb{F}_q Q_{4n} / J(\mathbb{F}_q Q_{4n}) \cong \mathbb{F}_q D_{2n} / J(\mathbb{F}_q D_{2n}) \cong \mathbb{F}_q \oplus \bigoplus_{m \mid n, \; m > 1} M(2, \mathbb{F}_q e_m) \frac{e(m)}{e_m}
\]

where

\[
e_m = \begin{cases} 
d_m / 2 & \text{if } d_m \text{ is even and } q^{d_m / 2} \equiv -1 \mod m \\
d_m & \text{otherwise}
\end{cases}
\]

and \( d_m = \text{ord}_m(q) \).

**Proof.** To distinguish the elements of \( D_{2N} \) from those of \( D_{2n} \), let \( D_{2N} \) be presented by

\[
\langle A, B \mid A^N, B^2, B^{-1} AB = A^{-1} \rangle
\]

and \( D_{2n} \) by

\[
\langle a, b \mid a^n, b^2, b^{-1} ab = a^{-1} \rangle
\]
From [14], it is known that
\[ F_q D_{2n} / J(F_q D_{2n}) \cong F_q \oplus \bigoplus_{m \mid n, m > 1} M(2, F_{q^{m}})^{(q^{m})/2} \]
Now
\[ \Delta(D_{2N}, \langle A^n \rangle) = \Delta(\langle A^n \rangle) F_q D_{2N} = J(F_q \langle A^n \rangle) F_q D_{2N} \subseteq J(F_q D_{2N}) \]
showing that \( dim_{F_q} J(F_q D_{2N}) \geq 2N - 2n \).

Since \( D_{2N}/\langle A^n \rangle \cong D_{2n} \), there exists an onto \( F_q \)-algebra homomorphism
\[ \phi : F_q D_{2N} \rightarrow F_q D_{2n} / J(F_q D_{2n}) \]
given by the assignment \( A \mapsto a + J(F_q D_{2N}) \), \( \beta \mapsto b + J(F_q D_{2N}) \) whence \( J(F_q D_{2N}) \subseteq ker \phi \) and
\[ dim_{F_q} J(F_q D_{2N}) \leq 2N - (2n - 1) = 2N - 2n + 1 \]
But there is only one 1-dimensional representation of \( D_{2N} \) over \( F_q \). This proves that \( dim_{F_q} J(F_q D_{2N}) = 2N - 2n + 1 \) and \( J(F_q D_{2N}) = ker \phi \) giving
\[ F_q D_{2N} / J(F_q D_{2N}) \cong F_q D_{2n} / J(F_q D_{2n}) \]
The decomposition of \( F_q Q_{4N} / J(F_q Q_{4N}) \) can be obtained by working on parallel lines.

\[ \square \]

**Theorem 2.6.** If \( n \) is odd, then
\[ U(F_q D_{4n}) \cong C_2^{(2n+1)k} \times \left( C_{q-1} \times \prod_{m \mid n, m > 1} GL(2, F_{q^{m}})^{(q^{m})/2} \right) \]
where
\[ e_m = \begin{cases} d_m/2 & \text{if } d_m \text{ is even and } q^{d_m/2} \equiv -1 \mod m \\ d_m & \text{otherwise} \end{cases} \]
and \( d_m = ord_m(q) \).

**Proof.** Let \( D_{4n} = \langle \alpha, \beta | \alpha^{2n}, \beta^2, \beta^{-1} \alpha \beta = \alpha^{-1} \rangle \) and \( X = 1 + \alpha^n \). Then \( \{ X, \alpha X, \cdots, \alpha^{n-1}X, \beta X, \beta \alpha X, \cdots, \beta \alpha^{n-1}X \} \) is a basis of \( \Delta(D_{4n}, \langle \alpha^n \rangle) \).

Observe that any \( W \in \Delta(D_{4n}, \langle \alpha^n \rangle) \) is expressible as
\[ W = (A_1 + A_2 \alpha + \cdots + A_n \alpha^{n-1} + A_{n+1} \beta + A_{n+2} \beta \alpha + \cdots + A_{2n} \beta \alpha^{n-1})X \]
for some \( A_i \in F_q \) so that
\[ W^2 = (A_1 + A_2 \alpha + \cdots + A_n \alpha^{n-1} + A_{n+1} \beta + A_{n+2} \beta \alpha + \cdots + A_{2n} \beta \alpha^{n-1})^2 (1 + \alpha^n)^2 = 0 \]
That is, \( 1 + \Delta(D_{4n}, \langle \alpha^n \rangle) \cong C_2^{2nk} \).

The \( F_q \)-algebra homomorphism
\[ \theta : F_q D_{4n} \rightarrow F_q D_{2n} / J(F_q D_{2n}) \]
given by the assignment
\[ \alpha \mapsto a + J(\mathbb{F}_q D_{2n}), \quad \beta \mapsto b + J(\mathbb{F}_q D_{2n}) \]
is onto and it is known that \( \mathcal{D}_{2n} \in J(\mathbb{F}_q D_{2n}) \). Thus if \( B = (1 + a + \cdots + a^{n-1})(1 + \beta) \), then \( \theta(B) = 0 + J(\mathbb{F}_q D_{2n}) \) showing that \( B \in \ker \theta = J(\mathbb{F}_q D_{4n}) \). In fact, \( J(\mathbb{F}_q D_{4n}) = \Delta(D_{4n}, \langle \alpha^n \rangle) \oplus \mathbb{F}_q B \) as a vector space over \( \mathbb{F}_q \).

Since
\[
B^2 = ((1 + \beta)(1 + \alpha + \cdots + \alpha^{n-1}))^2
\]
\[
= (1 + \beta)(1 + \beta \alpha^{n+1})(1 + \alpha + \cdots + \alpha^{n-1})^2
\]
\[
= (1 + \beta + \alpha^{n+1} + \beta \alpha^{n+1})(1 + \alpha^2 + \cdots + \alpha^{2n-2})
\]
\[
= 1 + \alpha^2 + \cdots + \alpha^{2n-2} + \beta + \beta \alpha^2 + \cdots + \beta \alpha^{2n-2} + \alpha^{n+1} + \cdots + \alpha^{n+3} + \cdots + \alpha^{3n+1} + \beta \alpha^{n+1} + \beta \alpha^{n+3} + \cdots + \beta \alpha^{3n-1}
\]
\[= 0 \text{ because } n \text{ is odd and } \alpha^{2n} = 1. \]

and
\[
X B = (1 + \alpha^n)(1 + \alpha + \cdots + \alpha^{n-1})(1 + \beta) = \hat{\alpha}(1 + \beta),
\]
we find that \( WB = BW \) so that
\[
V = 1 + J(\mathbb{F}_q D_{4n})
\]
\[
= 1 + \Delta(D_{4n}, \langle \alpha^n \rangle) \times \{ 1 + \eta B \mid \eta \in \mathbb{F}_q \}
\]
\[\cong \mathcal{C}^{(2n+1)k} \]
This completes the proof.

A group \( G \) is said to be the general product of its subgroups \( L \) and \( M \) if
\[ G = LM, \quad L \cap M = \{1\} \]

In this case, we write \( G = L \circ M \).

In the subsequent theorem, it is established that \( 1 + J(\mathbb{F}_q Q_{4n}) \) is a general product of two of its proper subgroups. As a consequence, the structure of \( \mathcal{U}(\mathbb{F}_q Q_{4n}) \) is obtained.

**Lemma 2.7.** Let \( G \) be a finite abelian \( p \)-group, \( G^{p^j} = \{ x^{p^j} \mid x \in G \} \) and \( p^{m_i} = |G^{p^i}| \). If \( G \cong \prod_{i=1}^{k} C_{p^{m_i}} \), then
\[ n_i = m_{i-1} - 2m_i + m_{i+1} \forall 1 \leq i \leq k \]

**Theorem 2.8.** If \( n \) is odd, then
\[ \mathcal{U}(\mathbb{F}_q Q_{4n}) \cong \left( C_2^{(2n-2)k} \circ \left( C_2^k \times C_4^k \right) \right) \times \left( C_{q-1} \times \prod_{m|n, \, m > 1} GL(2, \mathbb{F}_{q^m})^{\frac{x(m)}{2e_m}} \right) \]
where \( e_m \) as in Theorem 2.6.

**Proof.** Let \( Q_{4n} \) be presented by
\[ \langle C, \quad D \mid C^{2n}, \quad D^2 = C^n, \quad D^{-1}CD = C^{-1} \rangle \]
Let $U = (1 + D)(1 + C + \cdots + C^{n-1})$. Then via similar arguments as in the previous theorem, $U \in J(\mathbb{F}_q Q_{4n})$ and

$$U^2 = (1 + D)(1 + C^n) + D\hat{C} \left( 1 + C + \cdots + C^{n-1} \right) = (1 + D)\hat{C} + D\hat{C} = \hat{C}$$

Notice that if $Y = 1 + C^n$, then \{ $Y, CY, \cdots, C^{n-2}Y, DY, DCY, \cdots, DC^{n-2}Y, \hat{C}, (1 + D)\hat{C}, U$ \} is a basis of $J(\mathbb{F}_q Q_{4n})$ over $\mathbb{F}_q$.

Since $Y \in Z(\mathbb{F}_q Q_{4n})$ and $Y^2 = 0$, it is therefore evident that

$$H = \left\{ 1 + \left( \sum_{i=0}^{n-2} A_i C^i + \sum_{i=0}^{n-2} B_i D C^i \right) \bigg| A_i, B_i \in \mathbb{F}_q \right\} \leq 1 + J(\mathbb{F}_q Q_{4n})$$

and $H \cong C_2^{2(n-2)}k$. Also $K = \left\{ 1 + A_1 C + A_2 \hat{C} + A_3 (1 + D)\hat{C} \bigg| A_i \in \mathbb{F}_q \right\} \leq 1 + J(\mathbb{F}_q Q_{4n})$ and by Lemma 2.7 we find $K \cong C_2^k \times C_4^k$.

Since $H \cap K = \{ 1 \}$ and $|1 + J(\mathbb{F}_q Q_{4n})| = |HK|$, it follows that $1 + J(\mathbb{F}_q Q_{4n}) = H \circ K$. This completes the proof.

\[ \square \]

References


