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LEFT ANNIHILATOR OF COMMUTATOR IDENTITY WITH GENERALIZED DERIVATIONS AND MULTILINEAR POLYNOMIALS IN PRIME RINGS

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Let $R$ be a prime ring of characteristic different from 2 with Utumi quotient ring $U$ and extended centroid $C$, $F$ a nonzero generalized derivation of $R$, $I$ an ideal of $R$, and $f(x_1, \ldots, x_n)$ a multilinear polynomial over $C$ which is not central valued on $R$. If $0 \neq a \in R$ such that

$$a[F(u)v, F(v)u] = 0$$

for all $u, v \in f(I)$, where $f(I)$ is the set of all evaluations of $f(x_1, \ldots, x_n)$ in $I$, then there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$ and one of the following statements holds:

1. $f(x_1, \ldots, x_n)^2$ is central valued on $R$;
2. $ab = 0$.

Key Words: Derivation; Extended centroid; Generalized derivation; Prime ring; Utumi quotient ring.

2010 Mathematics Subject Classification: 16W25; 16N60.

1. INTRODUCTION

Throughout this article, $R$ always denotes an associative prime ring with extended centroid $C$ and $U$ its Utumi quotient ring. By $d$, we mean a derivation of $R$. The Lie commutator of $x$ and $y$ is denoted by $[x, y]$ and defined by $[x, y] = xy - yx$ for $x, y \in R$. Posner [20] proved that, if $a \in R$ such that $ad(x) = 0$ for all $x \in R$ or $d(x)a = 0$ for all $x \in R$, then either $a = 0$ or $d = 0$. A particular result of Dhara [10] states that, if $a \in R$ such that $ad(u)u = 0$ for all $u$ in a noncentral Lie ideal of $R$, then either $a = 0$ or $d = 0$. It is natural to investigate situations replacing derivation with generalized derivation. An additive mapping $F : R \to R$ is called a generalized derivation if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$.

In [21], Rania proved that, if $a \in R$ such that $aF(u)u = 0$ for all $u$ in a noncentral Lie ideal of $R$, then either $a = 0$ or $F(x) = bx$ for all $x \in R$, for
some \( b \in U \) with \( ab = 0 \). Note that a noncentral Lie ideal of \( R \) contains all the commutators \([x_1, x_2]\) for \( x_1, x_2 \) in some nonzero ideal of \( R \) except when \( \text{char}(R) = 2 \) and \( R \) satisfies \( s_4 \) (see [3]). So it is natural to consider the situation when \( aF(x)x = 0 \) for all commutators \( x = [x_1, x_2] \) or in the more general case when \( x = f(x_1, \ldots, x_n) \), where \( f(x_1, \ldots, x_n) \) is a multilinear polynomial over \( C \).

Recently, in [2] Nurcan and Demir proved the following result.

Let \( K \) be a commutative ring with unity, \( R \) be a prime \( K \)-algebra of characteristic different from 2 with right Utumi quotient ring \( U \), and extended centroid \( C \). Suppose that \( F \) a nonzero generalized derivation of \( R \), \( f(x_1, \ldots, x_n) \) a multilinear polynomial over \( K \) which is not central valued on \( R \) and \( a \in R \) a fixed nonzero element. If \( aF(f(x_1, \ldots, x_n))f(x_1, \ldots, x_n) = 0 \) for all \( x_1, \ldots, x_n \in R \), there exists \( b \in U \) such that \( F(x) = bx \) for all \( x \in R \) and \( ab = 0 \).

Recently, Ali, De Filippis, and Shujat [1] proved the following theorem.

Let \( R \) be a prime ring of characteristic different from 2, \( Z(R) \) the center of \( R \), \( U \) the two-sided Utumi quotient ring of \( R \), \( f(x_1, \ldots, x_n) \) a noncentral multilinear polynomial over \( K \), and \( F \) a nonzero generalized derivation of \( R \). Denote \( f(R) \) the set of all evaluations of the polynomial \( f(x_1, \ldots, x_n) \) in \( R \). If \( [F(u)u, F(v)v] = 0 \), for any \( u, v \in f(R) \), then there exists \( c \in U \) such that \( F(x) = cx \), for all \( x \in R \), and one of the following holds: (1) \( f(x_1, \ldots, x_n)^2 \) is central valued on \( R \); and (2) \( R \) satisfies \( s_4 \), the standard identity of degree 4.

In the present article, our main objective is to investigate the situation with left annihilator condition, that is, \( a[F(u)u, F(v)v] = 0 \), for any \( u, v \in f(R) \), where \( F \) is a generalized derivation of \( R \) and \( 0 \neq a \in R \). More precisely, we prove the following theorem.

**Theorem 1.1.** Let \( R \) be a prime ring of characteristic different from 2 with Utumi quotient ring \( U \) and extended centroid \( C \). Suppose that \( F \) a nonzero generalized derivation of \( R \), \( I \) an ideal of \( R \), and \( f(x_1, \ldots, x_n) \) a multilinear polynomial over \( C \) which is not central valued on \( R \). If \( 0 \neq a \in R \) such that

\[
a[F(u)u, F(v)v] = 0
\]

for all \( u, v \in f(I) \), then there exists \( b \in U \) such that \( F(x) = bx \) for all \( x \in R \), and one of the following statements holds:

1. \( f(x_1, \ldots, x_n)^2 \) is central valued on \( R \);
2. \( ab = 0 \).

**2. MAIN RESULTS**

We investigate the situation first when \( F \) is an inner generalized derivation of \( R \). Let \( F(x) = bx + xc \) for all \( x \in R \), for some \( b, c \in U \). Then \( a[F(u)u, F(v)v] = 0 \) for any \( u, v \in f(R) \) implies \( (ab^2 + aucu)(bv^2 + vcv) - (abv^2 + avev)(bu^2 + ucu) = 0 \), i.e., \( (pu^2 + aucu)(bv^2 + vcv) - (pv^2 + avev)(bu^2 + ucu) = 0 \) for any \( u, v \in f(R) \), where \( p = ab \). First we study this situation in a matrix ring.
We need the following lemma.

Lemma 2.1 ([9, Lemma 1]). Let $C$ be an infinite field and $m \geq 2$. If $A_1, \ldots, A_k$ are not scalar matrices in $M_m(C)$, then there exists some invertible matrix $P \in M_m(C)$ such that any matrices $PA_i P^{-1}, \ldots, PA_k P^{-1}$ have all nonzero entries.

Proposition 2.2. Let $R = M_m(C)$ be the ring of all $m \times m$ matrices over the infinite field $C$, $f(x_1, \ldots, x_n)$ a noncentral multilinear polynomial over $C$, and $a, b, c, p \in R$. If $(pf(r_1^2 + af(r)cf(r))(bf(s)^2 + f(s)cf(s)) - (pf(s)^2 + af(s)cf(s))(bf(r)^2 + f(r)cf(r)) = 0$ for all $r = (r_1, \ldots, r_n) \in R^n$ and $s = (s_1, \ldots, s_n) \in R^n$, then either $a$ or $c$ are central.

Proof. By our assumption, $R$ satisfies the generalized identity

\begin{align*}
(pf(r_1, \ldots, r_n)^2 + af(r_1, \ldots, r_n)cf(r_1, \ldots, r_n)) \\
(bf(s_1, \ldots, s_n)^2 + f(s_1, \ldots, s_n)cf(s_1, \ldots, s_n)) \\
- (pf(s_1, \ldots, s_n)^2 + af(s_1, \ldots, s_n)cf(s_1, \ldots, s_n)) \\
(bf(r_1, \ldots, r_n)^2 + f(r_1, \ldots, r_n)cf(r_1, \ldots, r_n)) = 0. \tag{1}
\end{align*}

We assume first that $a \notin Z(R)$ and $c \notin Z(R)$. Now we shall show that this case leads to a contradiction.

Since $a \notin Z(R)$ and $c \notin Z(R)$, by Lemma 2.1 there exists a $C$-automorphism $\phi$ of $M_m(C)$ such that $a_1 = \phi(a)$, $c_1 = \phi(c)$ have all nonzero entries. Clearly $a_1$, $c_1$, $b_1 = \phi(b)$ and $p_1 = \phi(p)$ must satisfy the condition (1). Without loss of generality, we may replace $a, b, c, p$ with $a_1, b_1, c_1, p_1$, respectively.

Here $e_{kl}$ denotes the matrix whose $(k, l)$-entry is 1 and all other entries are zero. Since $f(x_1, \ldots, x_n)$ is not central, by [17] (see also [18]), there exist $u_1, \ldots, u_n \in M_m(C)$ and $\gamma \in C - \{0\}$ such that $f(u_1, \ldots, u_n) = \gamma e_{kl}$, with $k \neq l$. Moreover, since the set $\{f(r_1, \ldots, r_n) : r_1, \ldots, r_n \in M_m(C)\}$ is invariant under the action of all $C$-automorphisms of $M_m(C)$, then for any $i \neq j$ there exist $r_1, \ldots, r_n \in M_m(C)$ such that $f(r_1, \ldots, r_n) = e_{ij}$ and there exist $s_1, \ldots, s_n \in M_m(C)$ such that $f(s_1, \ldots, s_n) = e_{ij}$. Hence by (1), we have

\begin{equation}
(ae_{ij}ce_{ij})(e_{ij}ce_{ij}) - (ae_{ij}ce_{ij})(e_{ij}ce_{ij}) = 0. \tag{2}
\end{equation}

Left multiplying by $e_{ij}$, we obtain $(a e_{ij} - a_{ij}e_{ij})c_{ij}e_{ij} = 0$. Now again right multiplying by $e_{ij}$, we get $a_{ij}c_{ij}e_{ij}e_{ij} = 0$. This implies $a_{ij}c_{ij}e_{ij} = 0$. This is a contradiction, since $a$ and $c$ have all nonzero entries. Thus we conclude that either $a$ or $c$ are central.\hfill \Box

Proposition 2.3. Let $R = M_m(C)$ be the ring of all matrices over the field $C$ with char $\langle R \rangle \neq 2$, and let $f(x_1, \ldots, x_n)$ be a noncentral multilinear polynomial over $C$ and $a, b, c, p \in R$. If $(pf(r)^2 + af(r)cf(r))(bf(s)^2 + f(s)cf(s)) - (pf(s)^2 + af(s)cf(s))(bf(r)^2 + f(r)cf(r)) = 0$ for all $r = (r_1, \ldots, r_n) \in R^n$ and $s = (s_1, \ldots, s_n) \in R^n$, then either $a$ or $c$ are central.
Proof. If one assumes that \( C \) is infinite, then the conclusions follow by Proposition 2.2.

Now let \( C \) be finite and \( K \) be an infinite field which is an extension of the field \( C \). Let \( \overline{R} = M_n(K) \cong R \otimes_K K \). Notice that the multilinear polynomial \( f(x_1, \ldots, x_n) \) is central-valued on \( R \) if and only if it is central-valued on \( \overline{R} \). Consider the generalized polynomial

\[
P(r_1, \ldots, r_n; s_1, \ldots, s_n) = (pf(r_1, \ldots, r_n)^2 + af(r_1, \ldots, r_n)c(f(r_1, \ldots, r_n)).
\]

which is a generalized polynomial identity for \( R \).

Moreover, it is a multihomogeneous of multidegree \((2, \ldots, 2)\) in the indeterminates \( r_1, \ldots, r_n, s_1, \ldots, s_n \).

Hence the complete linearization of \( P(r_1, \ldots, r_n; s_1, \ldots, s_n) \) is a multilinear generalized polynomial \( \Theta(r_1, \ldots, r_n, x_1, \ldots, x_n, s_1, \ldots, s_n, y_1, \ldots, y_n) \) in \( 4n \) indeterminates and moreover

\[
\Theta(r_1, \ldots, r_n, r_1, \ldots, r_n, s_1, \ldots, s_n, y_1, \ldots, y_n) = 2^n 2^n P(r_1, \ldots, r_n, s_1, \ldots, s_n).
\]

Clearly, the multilinear polynomial \( \Theta(r_1, \ldots, r_n, x_1, \ldots, x_n, s_1, \ldots, s_n, y_1, \ldots, y_n) \) is a generalized polynomial identity for \( R \) and \( \overline{R} \) too. Since \( \text{char}(C) \neq 2 \), we obtain \( P(r_1, \ldots, r_n, s_1, \ldots, s_n) = 0 \) for all \( r_1, \ldots, r_n, s_1, \ldots, s_n \in \overline{R} \), and then the conclusion follows from Proposition 2.2. \( \square \)

**Lemma 2.4.** Let \( R \) be a prime ring of characteristic different from 2 with Utumi quotient ring \( U \) and extended centroid \( C \), and let \( f(x_1, \ldots, x_n) \) be a multilinear polynomial over \( C \), which is not central valued on \( R \). Suppose that for some \( a, b, c, p \in U \), \( (pf(r)^2 + af(r)c(f(r)))(bf(s)^2 + cf(s)c(f(s))) = (pf(s)^2 + af(s)c(f(s))(bf(r)^2 + f(r)c(f(r))) = 0 \) for all \( r = (r_1, \ldots, r_n) \in R^n \) and \( s = (s_1, \ldots, s_n) \in R^n \). Then either \( a \) or \( c \) is central.

**Proof.** Let \( a \not\in C \) and \( c \not\in C \). By hypothesis, we have

\[
h(x_1, \ldots, x_n, y_1, \ldots, y_n) = (pf(x_1, \ldots, x_n)^2 + af(x_1, \ldots, x_n)c(f(x_1, \ldots, x_n))
\]

\[
(bf(y_1, \ldots, y_n)^2 + f(y_1, \ldots, y_n)c(f(y_1, \ldots, y_n)))
\]

\[
- (pf(y_1, \ldots, y_n)^2 + af(y_1, \ldots, y_n)c(f(y_1, \ldots, y_n)).
\]

\[
(bf(x_1, \ldots, x_n)^2 + f(x_1, \ldots, x_n)c(f(x_1, \ldots, x_n)) = 0 \quad (4)
\]

for all \( x_1, \ldots, x_n, y_1, \ldots, y_n \in R \). Since \( R \) and \( U \) satisfy same generalized polynomial identity (see [6]), \( U \) satisfies \( h(x_1, \ldots, x_n, y_1, \ldots, y_n) = 0 \). Suppose that \( h(x_1, \ldots, x_n, y_1, \ldots, y_n) \) is a trivial generalized polynomial identity (GPI) for \( U \).
Let \( T = U \ast_C C[x_1, x_2, \ldots, x_n, y_1, \ldots, y_n] \), the free product of \( U \) and \( C[x_1, \ldots, x_n, y_1, \ldots, y_n] \), the free \( C \)-algebra in noncommuting indeterminates \( x_1, x_2, \ldots, x_n, y_1, \ldots, y_n \). Then, \( h(x_1, \ldots, x_n, y_1, \ldots, y_n) \) is zero element in \( T = U \ast_C C[x_1, \ldots, x_n, y_1, \ldots, y_n] \). In the expansion of (4), we see that the term \( af(x_1, \ldots, x_n) cf(x_1, \ldots, x_n) f(y_1, \ldots, y_n) cf(y_1, \ldots, y_n) \) appears nontrivially, a contradiction.

Next suppose that \( h(x_1, \ldots, x_n) = 0 \) for all \( x_1, \ldots, x_n \in U \otimes_C \overline{C} \), where \( \overline{C} \) is the algebraic closure of \( C \). Since both \( U \) and \( U \otimes_C C \) are prime and centrally closed [11, Theorems 2.5 and 3.5], we may replace \( R \) by \( U \) or \( U \otimes_C \overline{C} \) according to \( C \) finite or infinite. Then \( R \) is centrally closed over \( C \) and \( h(x_1, \ldots, x_n) = 0 \) for all \( x_1, \ldots, x_n \in R \). By Martindale’s theorem [19], \( R \) is then a primitive ring with nonzero socle \( soc(R) \) and with \( C \) as its associated division ring. Then, by Jacobson’s theorem [13, p. 75], \( R \) is isomorphic to a dense ring of linear transformations of a vector space \( V \) over \( C \).

Assume first that \( V \) is finite dimensional over \( C \), that is, \( dim_C V = m \). By the density of \( R \), we have \( R \cong M_m(C) \). Since \( f(r_1, \ldots, r_n) \) is not central valued on \( R, \) \( R \) must be noncommutative and so \( m \geq 2 \). In this case, by Proposition 2.3, we get that either \( a \) or \( c \) is in \( C \), a contradiction.

If \( V \) is infinite dimensional over \( C \), then for any \( e^2 = e \in soc(R) \) we have \( eRe \cong M_k(C) \) with \( k = dim_C Ve \). Since \( a \) and \( c \) are not in \( C \), there exist \( h, h' \in soc(R) \) such that \( [a, h] \neq 0 \) and \( [c, h'] \neq 0 \). By Litoff’s Theorem [12], there exists idempotent \( e \in soc(R) \) such that \( ah, ha, ch', h'c, h, h' \in eRe \). We have \( eRe \cong M_k(C) \) with \( k = dim_C Ve \). Since \( R \) satisfies the generalized identity

\[
\begin{align*}
(e pf(ex_1 e, \ldots, ex_n e)^2 + af(ex_1 e, \ldots, ex_n e) cf(ex_1 e, \ldots, ex_n e)) \\
(bf(ey_1 e, \ldots, ey_n e)^2 + f(ey_1 e, \ldots, ey_n e) cf(ey_1 e, \ldots, ey_n e)) e \\
- e(pf(ey_1 e, \ldots, ey_n e)^2 + af(ey_1 e, \ldots, ey_n e) cf(ey_1 e, \ldots, ey_n e)) \\
(bf(ex_1 e, \ldots, ex_n e)^2 + f(ex_1 e, \ldots, ex_n e) cf(ex_1 e, \ldots, ex_n e)) e = 0,
\end{align*}
\]

the subring \( eRe \) satisfies

\[
\begin{align*}
(epef(x_1, \ldots, x_n)^2 + eaef(x_1, \ldots, x_n) ecef(x_1, \ldots, x_n)) \\
(ebef(y_1, \ldots, y_n)^2 + f(y_1, \ldots, y_n) ecef(y_1, \ldots, y_n)) \\
- (epef(y_1, \ldots, y_n)^2 + eaef(y_1, \ldots, y_n) ecef(y_1, \ldots, y_n)) \\
(ebef(x_1, \ldots, x_n)^2 + f(x_1, \ldots, x_n) ecef(x_1, \ldots, x_n)) = 0.
\end{align*}
\]

Then by the above finite dimensional case, either \( eae \) or \( ece \) are central elements of \( eRe \). Thus either \( ah = (eae)h = heae = ha \) or \( ch' = (ece)h' = h'(ece) = h'c \), a contradiction.

**Lemma 2.5.** Let \( R \) be a noncommutative prime ring with extended centroid \( C \) and \( char(R) \neq 2 \). If \( a, b \in U \). If \( 0 \neq a \in R \) such that \( a[b[x_1, x_2], b[y_1, y_2]] = 0 \) for all \( x_1, x_2, y_1, y_2 \in R \), then \( ab = 0 \).

**Proof.** By [6], \( a[b[x_1, x_2], b[y_1, y_2]] = 0 \) is a generalized polynomial identity for \( R \) and for \( U \).
If \( b \) is in \( C \), then we have \( ab^2[[x_1, x_2], [y_1, y_2]] = 0 \) for all \( x_1, x_2, y_1, y_2 \in R \). This implies either \( b = 0 \) or \( a[[x_1, x_2], [y_1, y_2]] = 0 \) for all \( x_1, x_2, y_1, y_2 \in R \). Assume that \( a[[x_1, x_2], [y_1, y_2]] = 0 \) for all \( x_1, x_2, y_1, y_2 \in R \). Then for \( w = [[x_1, x_2], [y_1, y_2]] \), we have \( aw = 0 \). Therefore, we can write \( a[[w, xa], [y, za]] = 0 \) for all \( x, y, z \in R \). Since \( aw = 0 \), it reduces to \( axaywxa = 0 \) for all \( x, y, z \in R \). Since \( R \) is prime and \( a \neq 0 \), we have \( w = 0 \), that is, \( [[x_1, x_2], [y_1, y_2]] = 0 \) for all \( x_1, x_2, y_1, y_2 \in R \). This is a polynomial identity, and hence there exists a field \( K \) such that \( R \subseteq M_q(F) \) with \( k > 1 \), and \( R \) and \( M_q(K) \) satisfy the same polynomial identity [15, Lemma 1].

By comparing (7) with (8), we get both \( \mathcal{C} \) linearly independent. In other words, we have

\[
\mathcal{C} \text{ linearly independent, as are } \mathcal{C} \text{ dependent, as are } \mathcal{C} \text{ is centrally closed over } R. \text{ Since } R \text{ is prime and } a \neq 0, \text{ we have } w = 0, \text{ that is, } [[x_1, x_2], [y_1, y_2]] = 0 \text{ for all } x_1, x_2, y_1, y_2 \in R. \text{ This is a polynomial identity, and hence there exists a field } K \text{ such that } R \subseteq M_q(F) \text{ with } k > 1, \text{ and } R \text{ and } M_q(K) \text{ satisfy the same polynomial identity [15, Lemma 1].}

Next we assume that \( b \not\in C \) and \( ab \neq 0 \). Then \( a[b[x_1, x_2], b[y_1, y_2]] = 0 \) is a nontrivial generalized polynomial identity for \( U \) and \( U \otimes_C C \). Since \( U \) and \( U \otimes_C C \) are both centrally closed [11], we may replace \( R \) by \( U \) or \( U \otimes_C C \) according as \( C \) is finite or infinite. Thus we may assume that \( R \) is centrally closed over \( C \) which is either finite or algebraically closed. By Martindale’s Theorem [19], \( R \) is a primitive ring having a nonzero socle with \( C \) as associated division ring. In this case, \( R \) is isomorphic to a dense ring of linear transformations on some vector space \( V \) over \( C \). Since \( R \) is not commutative, \( \dim_C V \geq 2 \). If for some \( v \in V \), \( v \) and \( bv \) are linearly \( C \)-independent, then by density, there exist \( x_1, x_2, y_1, y_2 \in R \) such that \( x_1v = 0, x_1v = bv, x_2v = 0, y_1v = 0 \). Thus \( \mathcal{C} \) is not commutative, as associated division ring. In this case, \( R \) is isomorphic to a dense ring of linear transformations on some vector space \( V \) over \( C \). Since \( R \) is not commutative, \( \dim_C V \geq 2 \). If for some \( v \in V \), \( v \) and \( bv \) are linearly \( C \)-independent, then by density, there exist \( x_1, x_2, y_1, y_2 \in R \) such that \( x_1v = 0, x_1v = bv, x_2v = 0, y_1v = 0 \). Then we obtain \( 0 = a[b[x_1, x_2], b[y_1, y_2]]v = abv. \) Of course, for any \( u \in V \), \( \{u, v\} \) linearly \( C \)-independent implies \( abu = 0. \) Since \( ab \neq 0 \), there exists \( u \in V \) such that \( abu \neq 0 \) and so \( \{w, v\} \) are linearly \( C \)-independent. Also \( ab(w + v) = abw \neq 0 \) and \( ab(w - v) = abw \neq 0. \) By the above argument, it follows that \( w \) and \( bw \) are linearly \( C \)-independent, as are \( \{w + v, b(w + v)\} \) \{w - v, b(w - v)\}. Therefore, there exist \( \alpha_w, \alpha_{w+v}, \alpha_{w-v} \in K \) such that

\[
bw = \alpha_w w, \quad b(w + v) = \alpha_{w+v} (w + v), \quad b(w - v) = \alpha_{w-v} (w - v).
\]

In other words, we have

\[
\alpha_w w + bv = \alpha_{w+v} w + \alpha_{w+v} v, \tag{7}
\]

and

\[
\alpha_w w - bv = \alpha_{w-v} w - \alpha_{w-v} v. \tag{8}
\]

By comparing (7) with (8), we get both

\[
(2\alpha_w - \alpha_{w+v} - \alpha_{w-v})w + (\alpha_{w-v} - \alpha_{w+v})v = 0 \tag{9}
\]

and

\[
2bv = (\alpha_{w+v} - \alpha_{w-v})w + (\alpha_{w+v} + \alpha_{w-v})v. \tag{10}
\]

By (9), and since \( \{w, v\} \) are \( C \)-independent and \( \text{char } (R) \neq 2 \), we have \( \alpha_w = \alpha_{w+v} = \alpha_{w-v}. \) Thus by (10) it follows \( 2bv = 2\alpha_w v. \) This leads a contradiction with the fact that \( \{v, bv\} \) is linear \( C \)-independent.
In light of this, we may assume that for any \( v \in V \) there exists a suitable \( z_v \in C \) such that \( bv = z_vv \), and standard argument shows that there is \( z \in C \) such that \( bv = zv \) for all \( v \in V \). Now let \( r \in R \), \( v \in V \). Since \( bv = zv \),

\[
[b, r]v = (br)v - (rb)v = b(rv) - r(bv) = z(rv) - r(zv) = 0.
\]

Thus \( [b, r]v = 0 \) for all \( v \in V \), i.e., \([b, r]V = 0\). Since \([b, r]\) acts faithfully as a linear transformation on the vector space \( V \), \([b, r] = 0\) for all \( r \in R \). Therefore, \( b \in Z(R) \), a contradiction.

**Lemma 2.6.** Let \( R \) be a prime ring with extended centroid \( C \) and \( \text{char} (R) \neq 2 \), \( a, b \in U \), \( p(x_1, \ldots, x_n) \) be any polynomial over \( C \), which is not central valued on \( R \). If \( a \notin C \) such that \( [a[p(r), b(p(s))] = 0 \) for all \( r = (r_1, \ldots, r_n) \in R^n \) and \( s = (s_1, \ldots, s_n) \in R^n \), then \( ab = 0 \).

**Proof.** Since \( p(x_1, \ldots, x_n) \) is any polynomial over \( C \), which is not central valued on \( R \), \( R \) must be noncommutative. Let \( G \) be the additive subgroup of \( R \) generated by the set \( S = \{p(x_1, \ldots, x_n)|x_1, \ldots, x_n \in R \} \). Then \( S \neq \{0\} \), since \( p(x_1, \ldots, x_n) \) is noncentral valued on \( R \). By our assumption, we get \( a[bx, by] = 0 \) for any \( x, y \in G \). By [7], either \( G \subseteq Z(R) \) or \( \text{char} (R) = 2 \) and \( R \) satisfies \( s_4 \), except when \( G \) contains a noncentral Lie ideal \( L \) of \( R \). Since \( p(x_1, \ldots, x_n) \) is not central valued on \( R \), the first case cannot occur. Since \( \text{char} (R) \neq 2 \), second case also cannot occur. Thus \( G \) contains a noncentral Lie ideal \( L \) of \( R \). By [3, Lemma 1], there exists a noncentral two sided ideal \( I \) of \( R \) such that \( [I, R] \subseteq L \). In particular, \( a[b[x_1, x_2], b[y_1, y_2]] = 0 \) for all \( x_1, x_2, y_1, y_2 \in I \). By [6], \( a[b[x_1, x_2], b[y_1, y_2]] = 0 \) is a generalized polynomial identity for \( R \) and for \( U \). Then by Lemma 2.5, we have \( ab = 0 \), as desired.

**Lemma 2.7** ([4]). Let \( R \) be a prime ring of characteristic different from 2, with Utumi quotient ring \( U \) and extended centroid \( C \), \( \delta \) a nonzero derivation of \( R \), \( G \) a nonzero generalized derivation of \( R \), and \( f(x_1, \ldots, x_n) \) a noncentral multilinear polynomial over \( C \). If \( \delta(G(f(x_1, \ldots, x_n)f(x_1, \ldots, x_n))) = 0 \) for all \( x_1, \ldots, x_n \in R \), then \( f(x_1, \ldots, x_n)^2 \) is central-valued on \( R \). Moreover, there exists \( a \in U \) such that \( G(x) = ax \) for all \( x \in R \) and \( \delta \) is an inner derivation of \( R \) such that \( \delta(a) = 0 \).

**Lemma 2.8.** Let \( R \) be a prime ring of characteristic different from 2 with Utumi quotient ring \( U \) and extended centroid \( C \), and let \( f(x_1, \ldots, x_n) \) be a multilinear polynomial over \( C \), which is not central valued on \( R \) and \( 0 \neq a \in R \). Suppose that for some \( b, c \in U \), \( F(x) = bx + xc \) for all \( x \in R \) is an inner generalized derivation of \( R \) such that \( a[F(u)u, F(v)v] = 0 \) for all \( u, v \in f(R) \). Then \( F(x) = (b + c)x \) for all \( x \in R \), and one of the following statements holds:

1. \( f(x_1, \ldots, x_n)^2 \) is central valued on \( R \);
2. \( a(b + c) = 0 \).

**Proof.** By hypothesis, we have

\[
a(bf(r)^2 + f(r)c(f(r))(bf(s)^2 + f(s)c(f(s)))
= -a(bf(s)^2 + f(s)c(f(s))(bf(r)^2 + f(r)c(f(r))) = 0;
\]

\[
(12)
\]
that is,

\[(abf(r)^2 + af(r)cf(r))(bf(s)^2 + f(s)cf(s)) - (abf(s)^2 + af(s)cf(s))(bf(r)^2 + f(r)cf(r)) = 0\]  

(13)

for all \( r = (r_1, \ldots, r_n) \in R^n \) and \( s = (s_1, \ldots, s_n) \in R^n \). Then by Proposition 2.3, either \( a \in C \) or \( c \in C \). If \( 0 \neq a \in C \), then \( a[F(u)u, F(v)v] = 0 \) for all \( u, v \in f(R) \) yields \( [F(u)u, F(v)v] = 0 \) for all \( u, v \in f(R) \). Then by Lemma 2.7, one of the following statements holds: (i) \( F(u)u \in C \) for all \( u \in f(R) \) (in this case by [8], there exists a \( x \in C \) such that \( F(x) = xx \) for all \( x \in R \) and \( f(x_1, \ldots, x_n)^2 \) is central valued on \( R \); and (ii) there exists \( p \in U \) such that \( F(x) = px \) for all \( x \in R \) and \( f(x_1, \ldots, x_n)^2 \) is central valued on \( R \).

In any case, we get our conclusion (1).

So assume that \( a \notin C \). Then \( c \in C \), and hence by our hypothesis, we have

\[a[(b + c)f(r)^2, (b + c)f(s)^2] = 0\]  

(14)

for all \( r = (r_1, \ldots, r_n) \in R^n \) and \( s = (s_1, \ldots, s_n) \in R^n \). Then by Lemma 2.6, one of the following statements holds: (1) \( f(x_1, \ldots, x_n)^2 \) is central valued on \( R \); (2) \( a(b + c) = 0 \). In both cases, \( F(x) = (b + c)x \) for all \( x \in R \). Thus the conclusions (1) and (2) are obtained.

\[\Box\]

**Proof of Theorem 1.1.** In [16, Theorem 3], Lee proved that every generalized derivation \( g \) on a dense right ideal of \( R \) can be uniquely extended to a generalized derivation of \( U \) and thus can be assumed to be defined on the whole \( U \) with the form \( g(x) = ax + d(x) \) for some \( a \in U \) and \( d \) is a derivation of \( U \). In light of this, we may assume that there exist \( b \in U \) and derivation \( d \) of \( U \) such that \( F(x) = bx + d(x) \). Since \( I, R, \) and \( U \) satisfy the same generalized polynomial identities (see [6]) as well as the same differential identities (see [17]), without loss of generality, we have

\[a[F(f(x_1, \ldots, x_n), f(x_1, \ldots, x_n)), f(y_1, \ldots, y_n)] = 0\]

for all \( x_1, \ldots, x_n, y_1, \ldots, y_n \in U \), where \( d \) is derivation on \( U \).

If \( F \) is inner generalized derivation of \( R \), then by Lemma 2.8 we obtain our conclusions. Thus we assume that \( F \) is not inner. Hence \( U \) satisfies

\[a[(bf(x_1, \ldots, x_n) + d(f(x_1, \ldots, x_n)))f(x_1, \ldots, x_n), (bf(y_1, \ldots, y_n)] = 0.\]  

(15)

By assumption, \( d \) cannot be an inner derivation of \( U \). Let \( f^d(x_1, \ldots, x_n) \) be the polynomial obtained from \( f(x_1, \ldots, x_n) \) replacing each coefficients \( x_a \) with \( d(x_a) \). Then we have

\[d(f(x_1, \ldots, x_n)) = f^d(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, d(x_i), \ldots, x_n),\]
and hence $U$ satisfies
\[
a \left[ (bf(x_1, \ldots, x_n) + f^d(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, d(x_i), \ldots, x_n))f(x_1, \ldots, x_n),
(bf(y_1, \ldots, y_n) + f^d(y_1, \ldots, y_n) + \sum_i f(y_1, \ldots, d(y_i), \ldots, y_n))f(y_1, \ldots, y_n) \right] = 0.
\]

Then by Kharchenko’s theorem [14], we have that $U$ satisfies
\[
a \left[ (bf(x_1, \ldots, x_n) + f^d(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, t_i, \ldots, x_n))f(x_1, \ldots, x_n),
(bf(y_1, \ldots, y_n) + f^d(y_1, \ldots, y_n) + \sum_i f(y_1, \ldots, s_i, \ldots, y_n))f(y_1, \ldots, y_n) \right] = 0.
\]

In particular, $U$ satisfies the blended component
\[
a \left[ \sum_i f(x_1, \ldots, t_i, \ldots, x_n)f(x_1, \ldots, x_n), \sum_i f(y_1, \ldots, s_i, \ldots, y_n)f(y_1, \ldots, y_n) \right] = 0.
\]

Since $f(x_1, \ldots, x_n)$ is noncentral valued on $R$, $R$ and so $U$ must be noncommutative. So we choose $q \in U$ such that $q \notin C$. Replacing $t$, with $[q, x_i]$ and $s$, with $[q, y_i]$ in the last relation, we get that $U$ satisfies
\[
a \left[ [q, f(x_1, \ldots, x_n)]f(x_1, \ldots, x_n), [q, f(y_1, \ldots, y_n)]f(y_1, \ldots, y_n) \right] = 0.
\]

Then by Lemma 2.8, we get the contradiction that $q \in C$.

**Corollary 2.9.** Let $R$ be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring $U$ and extended centroid $C$ and $F$ a nonzero generalized derivation of $R$. If $0 \neq a \in R$ such that
\[
a [F(x)x, F(y)y] = 0
\]
for all $x, y \in R$, then there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$ with $ab = 0$.

**Corollary 2.10.** Let $R$ be a prime ring of characteristic different from 2, $d$ a derivation of $R$, $I$ an ideal of $R$, and $f(x_1, \ldots, x_n)$ a multilinear polynomial over $C$ which is not central valued on $R$. If $0 \neq a \in R$ such that
\[
a [d(u)u, d(v)v] = 0
\]
for all $u, v \in f(I)$, then $d = 0$.

**Corollary 2.11.** Let $R$ be a prime ring of characteristic different from 2 with Utumi quotient ring $U$ and extended centroid $C$, $F$ a nonzero generalized derivation of $R$, $I$ an ideal of $R$, and $f(x_1, \ldots, x_n)$ a multilinear polynomial over $C$ which is not central
valued on $R$. If $0 \neq a \in R$ such that $aF(u)u = 0$ for all $u \in f(I)$, then there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$ with $ab = 0$.

**Proof.** By Theorem 1.1, there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$. Then by hypothesis, $abf(x_1, \ldots, x_n)^2 = 0$ for all $x_1, \ldots, x_n \in I$. Then by [5, Lemma 2 (I)], $ab = 0$ or $f(I) = 0$. Since $I$ and $R$ satisfy the same polynomial identities (see [6]), we have $f(R) = 0$, contradicting the fact that $f(x_1, \ldots, x_n)$ is not central valued on $R$.

□

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