THE STRUCTURE OF THE UNIT GROUP OF THE GROUP
ALGEBRA $\mathbb{F}S_5$ WHERE $\mathbb{F}$ IS A FINITE FIELD WITH
$\text{Char}(\mathbb{F}) = p > 5$

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Abstract

Let $\mathbb{F}_q$ denote a field having $q = p^n$ elements, where $p$ a prime, and let $S_5$ denote the symmetric group of degree 5. We give a complete structure of the unit group $U(\mathbb{F}_q S_5)$, $p > 5$.

Mathematics Subject Classification (2010): 16U60

Keywords: Group algebra, Wedderburn Decomposition, Unit Group

1 Introduction

Unit group of group algebra has been very fascinating and challenging. Recently, Gildea [2, 3, 4, 5, 6, 7, 8, 9, 10, 12] and Makhijani [10] have determined the unit group of certain finite group algebras. Makhijani [10] characterized the unit group of the finite group algebra of the alternating group $A_5$. In this paper we have characterized completely for $\mathbb{F}S_5$, $\text{char}\mathbb{F} > 5$.

Let $\mathbb{F}$ be a finite field of $\text{char}(\mathbb{F}) = p > 5$. We shall write $\mathbb{F} = \mathbb{F}_q = \mathbb{F}_{p^n}$, where $|\mathbb{F}| = p^n$. Since $|S_5| = 120 = 2^3 \cdot 3 \cdot 5$. The Jacobson radical $J(\mathbb{F}_q S_5) = 0$ by Maschke’s theorem. Also $\mathbb{F}S_5$ is a finit ring, hence Artinian. Further, by Wedderburn Decomposition

$$\mathbb{F}_q S_5 \cong M_{n_1}(D_1) \oplus M_{n_2}(D_2) \oplus \cdots \oplus M_{n_r}(D_r)$$

where $D_1, D_2, \cdots, D_r$ are finite dimensional division algebras over the finite field $\mathbb{F}$. Therefore $D_k = \mathbb{F}_{p^{n_k}}$, and

$$\mathbb{F}_q S_5 \cong M_{n_1}(\mathbb{F}_{p^{n_1}}) \oplus M_{n_2}(\mathbb{F}_{p^{n_2}}) \oplus \cdots \oplus M_{n_r}(\mathbb{F}_{p^{n_r}}).$$

Hence

$$U(\mathbb{F}_q S_5) \cong GL_{n_1}(\mathbb{F}_{p^{n_1}}) \times GL_{n_2}(\mathbb{F}_{p^{n_2}}) \times \cdots \times GL_{n_r}(\mathbb{F}_{p^{n_r}})$$

Thus the structure of the unit group $U(\mathbb{F}_q S_5)$ is determined completely once we
compute \( r, n_1, n_2, \ldots, n_r, m_1, m_2, \ldots m_r \). Finding \( r \) is easy. Clearly the centre, 
\[
Z(F_q S_5) \cong F_p^{m_1} \oplus F_p^{m_2} \oplus \cdots F_p^{m_r}
\]
where \( r \) is the number of distinct conjugacy classes of \( S_5 \). It can be seen easily that \( r = 7 \). We shall use the following presentation of the group \( S_5 \)
\[
S_5 = \langle a, b \mid a^2, b^5, (ab)^4, (bab^{-2}ab)^2 \rangle
\]
where \( a = (1, 2) \) and \( b = (1, 2, 3, 4, 5) \).

2 Preliminaries

Throughout the paper, \( F \) will denote a finite field with \( p^n \) elements and \( G \) will denote a finite group. We shall need the following results from Ferraz’s [1].

An element \( x \in G \) is called \( p \)-regular if \( p \nmid |x| \). Let \( s \) be the l.c.m. of the orders of the \( p \)-regular elements of \( G \), \( \theta \) a primitive \( s \)th root of unity over \( F \), the multiplicative group \( T_{G,F} \) is defined by
\[
T_{G,F} = \{ t | \theta \rightarrow \theta^t \text{ is an automorphism of } F(\theta) \text{ over } F \}.
\]
For \( p \)-regular elements \( g \), denote by \( \gamma_g \) the sum of all conjugates of \( g \) in \( G \). The cyclotomic \( F \)-class of \( \gamma_g \) is to be the set
\[
S_F(\gamma_g) = \{ \gamma_g | t \in T_{G,F} \}.
\]

**PROPOSITION 2.1**[1]. The number of simple components of \( \frac{FG}{J(F \theta)} \) is equal to the number of cyclotomic \( F \)-classes in \( G \).

**PROPOSITION 2.2**[1]. Suppose the Galois group \( Gal(F(\theta) : F) \) is cyclic and \( t \) be the number of cyclotomic \( F \)-classes in \( G \). If \( K_1, K_2, \ldots, K_t \) are the simple components of \( Z(\frac{FG}{J(F \theta)}) \) and \( S_1, S_2, \ldots, S_t \) are the cyclotomic \( F \)-classes of \( G \), then \( |S_i| = [K_i : F] \) with a suitable ordering of the indices.

From the above proposition it follows that
\[
Z(\frac{FG}{J(F \theta)}) \cong F_{p^{n_1}} \oplus F_{p^{n_2}} \oplus \cdots F_{p^{n_t}},
\]
where \( F_{n_i} \) denotes the unique field between \( F(\theta) \) and \( F \) such that \( [F_{n_i} : F] = n_i \).

We shall freely use Ferraz’s [1] result, since every finite extension of a finite field is a cyclic extension which gives \( Gal(F(\theta) : F) \) cyclic group. Again, \( T_{G,F_{n_i}} = \{ q' \text{ mods } 0 \leq i \leq d - 1 \} \), where \( d \) is the order of \( q \text{ mods} \). We shall use above form of \( T_{G,F} \) to compute order of the cyclotomic \( F \)-classes in \( G \), and the relation given in proposition 2.2.
3 The Structure of the unit group $U(F_qS_5), p > 5$

We need the following result given in [11]

**PROPOSITION 3.1.** [Prop. 3.6.11, [11]] Let $FG$ be a semisimple group algebra. If $G'$ denotes the commutator subgroup of $G$ then we can write

$$FG \cong FG_{G'} \oplus \Delta(G, G'),$$

where $FG_{G'} \cong F(G/G')$ is the sum of all commutative simple components of $FG$ and $\Delta(G, G')$ is the sum of all the others. Here $e_{G'} = \frac{G'}{|G'|}$ where $G'$ is the sum of all elements of $G'$.

**THEOREM 3.2.** Let $q = p^n, p > 5$ be a prime then, $F_qS_5$ is isomorphic to one of the following:

(i) $F_q \oplus F_q \oplus \mathbb{M}_2(F_q) \oplus \mathbb{M}_2(F_q) \oplus \mathbb{M}_2(F_q) \oplus \mathbb{M}_2(F_q)$

(ii) $F_q \oplus F_q \oplus \mathbb{M}_2(F_q) \oplus \mathbb{M}_2(F_q) \oplus \mathbb{M}_2(F_q) \oplus \mathbb{M}_2(F_q)$

(iii) $F_q \oplus F_q \oplus \mathbb{M}_2(F_q) \oplus \mathbb{M}_2(F_q) \oplus \mathbb{M}_2(F_q) \oplus \mathbb{M}_2(F_q)$

(iv) $F_q \oplus F_q \oplus \mathbb{M}_2(F_q) \oplus \mathbb{M}_2(F_q) \oplus \mathbb{M}_2(F_q) \oplus \mathbb{M}_2(F_q)$

**PROOF.** In our case $\frac{S_5}{S_5'} \cong C_2$ as $S_5' = A_5$. Also, $|S_5| = 2^3 \cdot 3 \cdot 5$ and therefore group algebra $F_qS_5$ is semi-simple as $p \nmid |G|$. Here $q = p^n, p > 5$.

By proposition 3.1

$$F_qS_5 = F_qS_5e_{S_5'} \oplus F_qS_5(S_5' - 1)$$

where $e_{S_5'} = e_{A_5} = \frac{\hat{G}}{|A_5|} = \frac{\sum_{\sigma \in A_5} \sigma}{60}$

$$F_qS_5e_{S_5'} = \text{sum of all commutative simple components of } F_qS_5.$$

However,

$$F_qS_5e_{S_5'} \cong F_q\left(\frac{S_5}{S_5'}\right) \cong F_q(C_2) \cong F_q \oplus F_q.$$ 

Therefore, by Wedderburn Decomposition Theorem

$$F_qS_5 \cong F_q \oplus F_q \oplus \sum_{i=1}^{5} \mathbb{M}_{n_i}(F_{q^{n_i}}),$$

for $n_i \geq 2$, with in all 7 simple components.

Now $p > 5$, so $q = p^n \equiv \pm 1 \mod 6$. Again $|S_5, (\gamma_g)| = 1$, for each $g \in S_5$. Hence each $k_i = 1$ in the Wedderburn Decomposition of $F_qS_5$. Since $p > 5$, therefore each element of $S_5$ is $p$–regular.

By dimension constraints, we have

$$\dim_{F_q}(F_qS_5) = 1 + 1 + n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2$$

$$120 = 1 + 1 + \sum_{k=1}^{5} n_k^2$$
\[ 118 = n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2 \]

Possible solutions of the equation are

- 2, 2, 2, 5, 9
- 2, 2, 5, 6, 7
- 2, 3, 4, 5, 8
- 4, 4, 5, 5, 6.

Hence \( \mathbb{F}_q S_5 \) is isomorphic to one of the following:

(i) \( \mathbb{F}_q \oplus \mathbb{F}_q \oplus M_2(\mathbb{F}_q) \oplus M_2(\mathbb{F}_q) \oplus M_3(\mathbb{F}_q) \oplus M_5(\mathbb{F}_q) \)

(ii) \( \mathbb{F}_q \oplus \mathbb{F}_q \oplus M_2(\mathbb{F}_q) \oplus M_2(\mathbb{F}_q) \oplus M_3(\mathbb{F}_q) \oplus M_4(\mathbb{F}_q) \oplus M_5(\mathbb{F}_q) \)

(iii) \( \mathbb{F}_q \oplus \mathbb{F}_q \oplus M_2(\mathbb{F}_q) \oplus M_3(\mathbb{F}_q) \oplus M_4(\mathbb{F}_q) \oplus M_5(\mathbb{F}_q) \)

(iv) \( \mathbb{F}_q \oplus \mathbb{F}_q \oplus M_4(\mathbb{F}_q) \oplus M_5(\mathbb{F}_q) \oplus M_5(\mathbb{F}_q) \oplus M_6(\mathbb{F}_q) \)

**COROLLARY 3.3** Let \( q = p^s \), where \( p > 5 \) be a prime, then \( U(\mathbb{F}_q S_5) \) is isomorphic to one of the following:

(i) \( U(\mathbb{F}_q S_5) \cong \mathbb{F}_q^* \times \mathbb{F}_q^* \times GL_2(\mathbb{F}_q) \times GL_2(\mathbb{F}_q) \times GL_3(\mathbb{F}_q) \times GL_5(\mathbb{F}_q) \times GL_6(\mathbb{F}_q) \)

(ii) \( U(\mathbb{F}_q S_5) \cong \mathbb{F}_q^* \times \mathbb{F}_q^* \times GL_2(\mathbb{F}_q) \times GL_2(\mathbb{F}_q) \times GL_5(\mathbb{F}_q) \times GL_6(\mathbb{F}_q) \times GL_7(\mathbb{F}_q) \)

(iii) \( U(\mathbb{F}_q S_5) \cong \mathbb{F}_q^* \times \mathbb{F}_q^* \times GL_2(\mathbb{F}_q) \times GL_3(\mathbb{F}_q) \times GL_4(\mathbb{F}_q) \times GL_5(\mathbb{F}_q) \times GL_7(\mathbb{F}_q) \)

(iv) \( U(\mathbb{F}_q S_5) \cong \mathbb{F}_q^* \times \mathbb{F}_q^* \times GL_2(\mathbb{F}_q) \times GL_4(\mathbb{F}_q) \times GL_5(\mathbb{F}_q) \times GL_5(\mathbb{F}_q) \times GL_6(\mathbb{F}_q) \)

**ACKNOWLEDGEMENTS** The first author is supported by Council of Scientific and Industrial Research (CSIR), New Delhi, Govt. of India, Under Grant No. F. No:09/086(1133)/2012-EMR-I.

**References**


