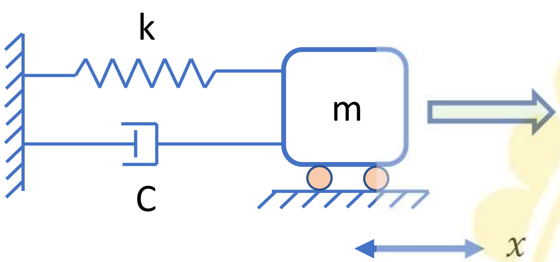
Lec 8

CONCEPT OF MECHANICAL IMPEDANCE



$$F = F_o e^{jwt}$$

Response: Displacement

$$x = x_o e^{j(wt-\varphi)}$$
$$= \left(x_o e^{-j\varphi}\right) \cdot e^{jwt}$$
$$= \bar{x}_o e^{jwt}$$

Phase lag of displacement w.r.t force

Include phase information

Harmonic Response: $\dot{x} = j\omega x = \omega x e^{\frac{\pi}{2}j}$

$$\ddot{x} = -\omega^2 x = \omega^2 x e^{\pi j}$$

Velocity leads displacement by $\frac{\pi}{2}$ & acceleration leads velocity by $\frac{\pi}{2}$

Equation of motion: $m\ddot{x} + c\dot{x} + kx = F = F_o e^{j\omega t}$

$$(-m\omega^2 + j\omega C + k)x = F$$

Effective dynamic stiffness: $\frac{F}{x} = -m\omega^2 + j\omega C + k$

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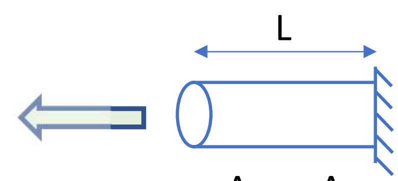
Special case: Negligible inertial force

$$(k_d)_{eff} = k + j\omega C$$
$$= (k + k^* j)$$
$$= \frac{A}{L}(\gamma + \gamma^* j)$$

$$F = F_o e^{jwt}$$



Define $\eta = \frac{\gamma^*}{\gamma}$ Mechanical loss factor

$$(k_d)_{eff} = \frac{A\gamma}{L}(1 + \eta j)$$

Complex Young Modulus

$$\bar{\gamma} = \gamma(1 + \eta j)$$

Under dynamic conditions

$$|\gamma|_{dynamic} = \gamma\sqrt{1 + \eta^2} > |\gamma|_{static}$$

Exact relation between $\eta, \xi, and\ C$:

$$k + j\omega C \equiv k + k\eta j$$

$$\omega C = k\eta$$

$$\eta = \left(\frac{C}{k}\right)\omega$$

Note: $\omega = \omega_N$

$$C = 2m\omega\,\xi$$

$C_c = Crtical\ damping$

$$[C] = \left[\frac{Ns}{m}\right] \qquad [k] = \left[\frac{N}{m}\right]$$

$$\omega^2 = \frac{k}{m}$$

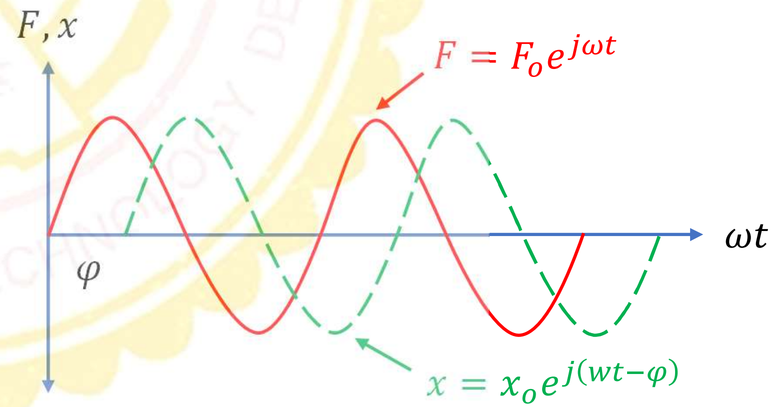
On Solving $\boldsymbol{\eta = 2\xi}$

$$F = (-m\omega^2 + j\omega C + k)x$$

If force, $F = F_o e^{j\omega t}$

Displacement, $x = x_o e^{j(wt-\varphi)}$

If no damping, $\varphi = 0$



Rayleigh Damping

$$C = \alpha m + \beta k$$

Mass multiplier (α) — Stiffness Multiplier (β)

How α & β related to ξ & η ?

Solution 1:

$$C = \alpha m + \beta k = \frac{k\eta}{\omega}$$

$$\Rightarrow \alpha = 0, \qquad \beta = \frac{\eta}{\omega} = \frac{2\xi}{\omega} = \frac{\xi}{\pi f}$$

Solution 2: Preferred in case of frequency range

$$\alpha m + \beta k = C = 2mw\xi$$

Infinite solutions

Divide both sides by $2m\omega$

$$\frac{\alpha m}{2m\omega} + \frac{\beta(m\omega^2)}{2m\omega} = \xi$$

$$\frac{\alpha}{2\omega} + \frac{\beta\omega}{2} = \xi$$

$$\frac{\alpha}{4\pi f} + \beta\pi f = \xi$$

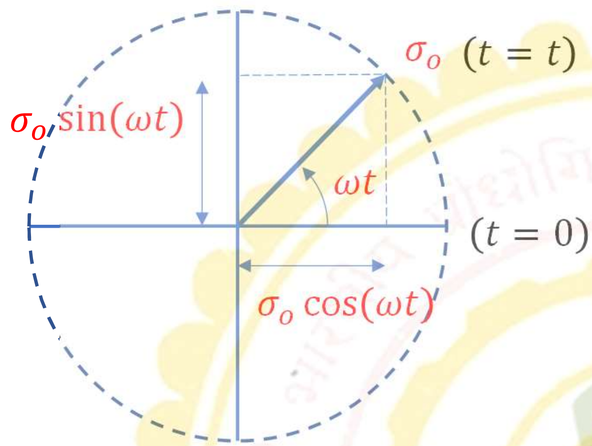
Let frequency range be (f_1, f_2)

Two possible equations

$$\frac{\alpha}{4\pi f_1} + \beta\pi f_1 = \xi \quad \dots (1)$$

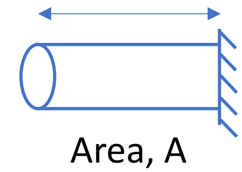
$$\frac{\alpha}{4\pi f_2} + \beta\pi f_2 = \xi \quad \dots (2)$$

Solve these 2 equations to get $(\alpha, \beta) \rightarrow$ Representation of range (f_1, f_2)



$$\sigma = \sigma_o e^{j\omega t}$$

$$= \sigma_o\left(\cos(\omega t) + j \cdot sin(\omega t)\right)$$



Either of the real or the imaginary component may be chosen

$$Strain, \qquad \epsilon = \frac{\sigma}{\gamma} = \frac{\sigma_o e^{j\omega t}}{\gamma(1+\eta j)} \times \frac{(1-\eta j)}{(1-\eta j)} \qquad \boldsymbol{\eta \ll 1}$$

$$= \frac{\sigma_o(1-\eta j)e^{j\omega t}}{\gamma(1+\eta^2)}$$

$$\approx \frac{\sigma_o}{\gamma} \cdot e^{-j\eta} \cdot e^{j\omega t}$$

$$\epsilon = \frac{\sigma_o}{\gamma} \cdot e^{j(\omega t - \eta)}$$

Phase lag

If $\sigma = \sigma_o \cos(\omega t)$ $\rightarrow$ Projection in X axis

Then $\epsilon = \epsilon_o \cos(\omega t - \eta)$

Alternatively, if $\sigma = \sigma_o \sin(\omega t)$, $\epsilon = \epsilon_o \sin(\omega t - \eta)$

If $\eta = 0$, σ & ϵ in phase: $\frac{\sigma_o}{\gamma(1+\eta^2)} \approx \frac{\sigma_o}{\gamma}$

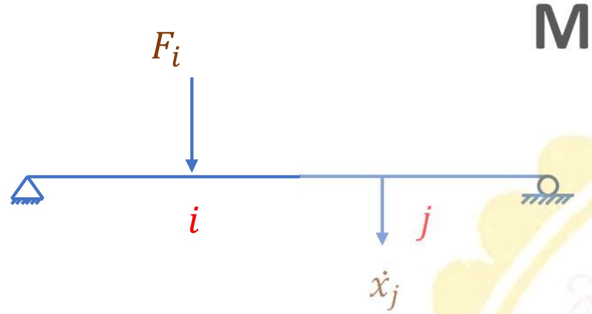
$$\epsilon = \frac{\sigma_o}{\gamma} \cdot e^{j(\omega t - \eta)}$$

$$= \left(\frac{\sigma_o}{\gamma} \cdot e^{-j\eta}\right) \cdot e^{j\omega t}$$

$$= (\epsilon_{o,r} + \epsilon_{o,i}) \cdot e^{j\omega t}$$

Phase information included in the magnitude part

Mechanical Impedance

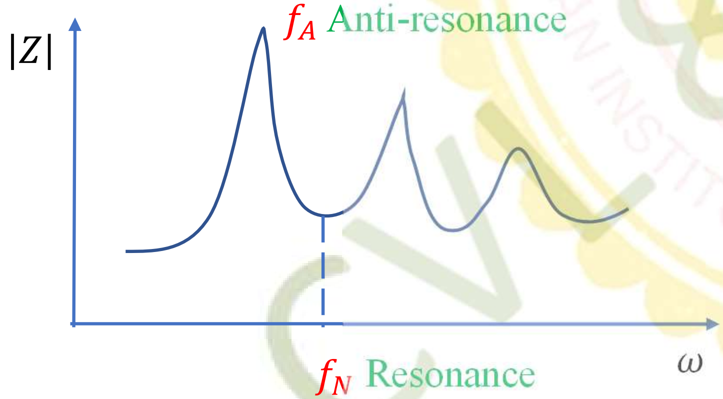


Measuring response

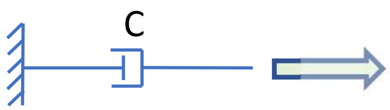
$$Z_{ij} = \frac{F_i}{\dot{x}_j}$$

Ratio of force to velocity

If $i = j$ then $Z_{ij} = \frac{F_i}{\dot{x}_j}$ → Ratio of force applied at one point in the structure to the resulting velocity at same point in the direction of force.

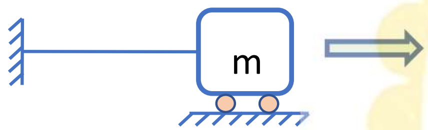


Similar to FRF



$F = F_o e^{jwt}$ $\qquad F = C\dot{x}$ $\qquad Z_{damper} = C$

$F = F_o e^{jwt}$ $\qquad F = kx = \dfrac{k\dot{x}}{j\omega} = \left(\dfrac{k}{j\omega}\right)\dot{x} = \left(-\dfrac{k}{\omega}j\right)\dot{x}$ $\qquad Z_{spring} = -\dfrac{k}{\omega}j$



$F = F_o e^{jwt}$ $\qquad F = m\ddot{x} = (mj\omega)\dot{x}$ $\qquad Z_{mass} = mj\omega$

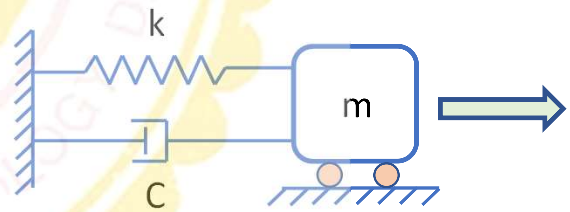
$$F_{damper} = (Z_{damper})\dot{x} = C\dot{x}$$

Lags $\dot{x}$ by $\frac{\pi}{2}$

$$F_{spring} = (Z_{spring})\dot{x} = -j\frac{k}{\omega}\dot{x} = \left(e^{-\frac{\pi}{2}j}\right)\frac{k}{\omega}\dot{x}$$

$$F_{mass} = (Z_{mass})\dot{x} = m\omega j\dot{x} = \left(e^{\frac{\pi}{2}j}\right)m\omega\dot{x}$$

Leads $\dot{x}$ by $\frac{\pi}{2}$



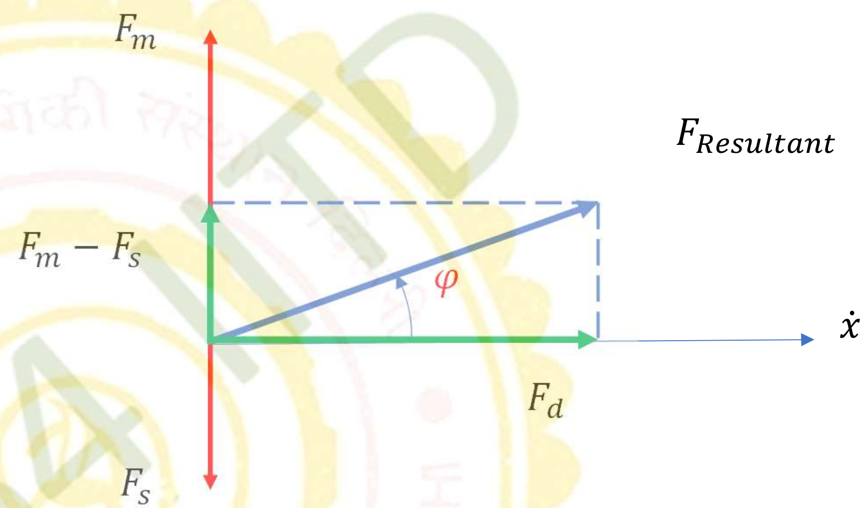
$F = F_o e^{jwt}$

$\dot{x} = \overline{\dot{x}_o} e^{j\omega t}$

Free Body Diagram

$$F_{Resultant} = \sqrt{F_d^2 + (F_m - F_s)^2}$$

$$= \sqrt{C^2 + \left(m\omega - \frac{k}{\omega}\right)^2}\ \dot{x}$$



$$\tan(\varphi) = \frac{F_m - F_s}{F_d}$$

$$= \frac{m\omega - \frac{k}{\omega}}{C}$$

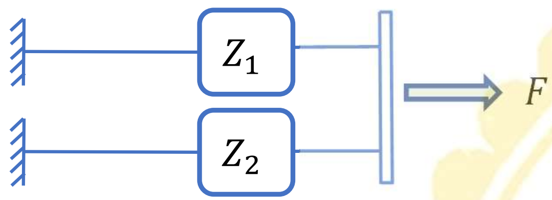
$$Z_{eff} = Z_{resultant} = \sqrt{C^2 + \left(m\omega - \frac{k}{\omega}\right)^2}$$

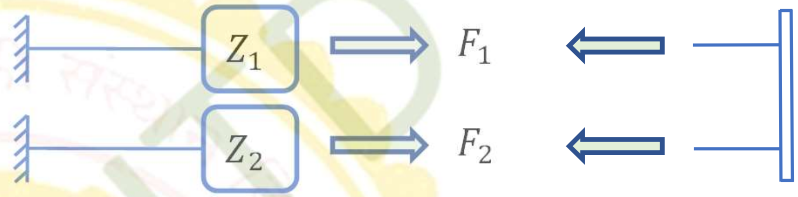
$$Phase\ Lag = \varphi = \tan^{-1}\left(\frac{m\omega - \frac{k}{\omega}}{C}\right)$$

$$\dot{x} = \left(\frac{F_o}{Z_{eff}}\right) \cdot e^{j(\omega t - \varphi)}$$

Single term representation of mass, stiffness and damping

In parallel:



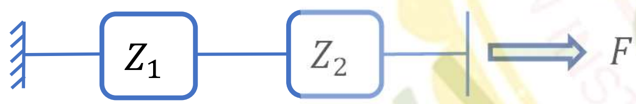


$F = F_1 + F_2 \qquad \dot{x} = \dot{x}_1 = \dot{x}_2$

$$F = F_1 + F_2 = Z_1\dot{x} + Z_2\dot{x} = (Z_1 + Z_2)\dot{x}$$

$$Z_{eq} = Z_1 + Z_2 + \cdots + Z_n = \sum_{i=1}^{N} Z_i$$

In series

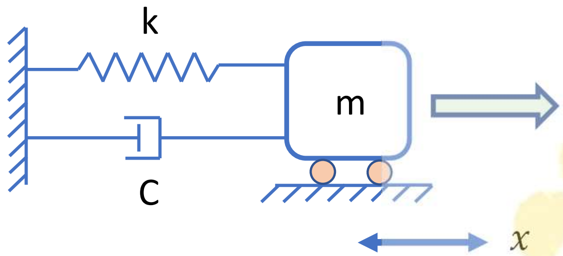


$F = F_1 = F_2 \qquad \dot{x} = \dot{x}_1 + \dot{x}_2$

$$\dot{x} = \dot{x}_1 + \dot{x}_2 = \frac{F_1}{Z_1} + \frac{F_2}{Z_2} = F\left(\frac{1}{Z_1} + \frac{1}{Z_2}\right)$$

$$\frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_n} = \sum_{i=1}^{N} \frac{1}{Z_i}$$

Classical mechanics representations



$F = F_o e^{jwt}$

$\dot{x}$ is same for all elements

$$F = m\ddot{x} + c\dot{x} + kx$$

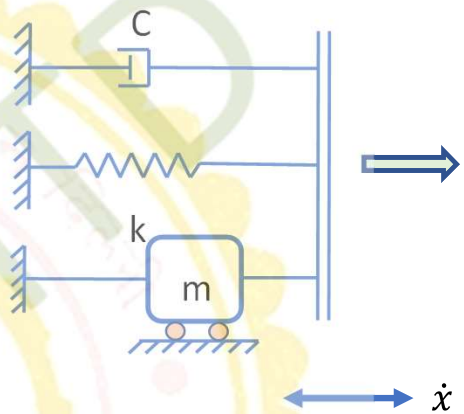
For a pure damper $C(k = m = 0)$, $\varphi = 0$ No phase Lag

$$Z_{eq} = \sum_{i=1}^{N} Z_i = C + m\omega j - \frac{k}{\omega} j$$

$$Z_{eq} = C + \left(m\omega - \frac{k}{\omega}\right) j$$

$$Z_{eq} = X + Yj$$

Impedance mechanics



$F = F_o e^{jwt}$

$\dot{x} = \dot{x}_o e^{j(\omega t - \varphi)}$

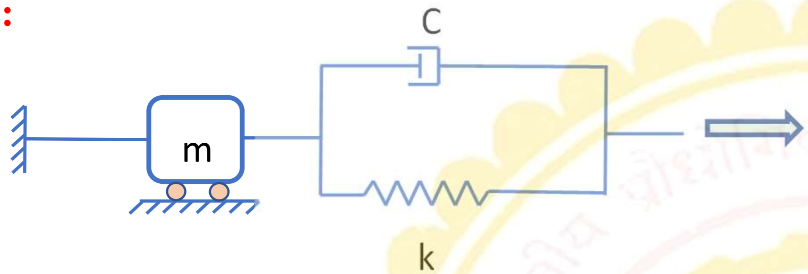
$$F = Z_{eq}\dot{x} = \left[C + \left(m\omega - \frac{k}{\omega}\right) j\right] \dot{x}$$

Equivalent to $(|Z|, \varphi)$ form

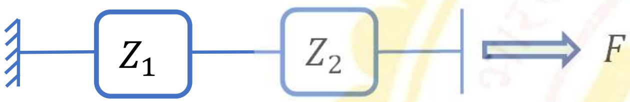
$$|Z| = \sqrt{C^2 + \left(m\omega - \frac{k}{\omega}\right)^2}$$

$$\varphi = tan^{-1}\left(\frac{m\omega - \frac{k}{\omega}}{C}\right)$$

Example:



$$F = F_o e^{jwt}$$



$$Z_1 = m\omega j$$

$$Z_2 = C - \frac{k}{\omega} j$$

$$Z_{eq} = \frac{Z_1 Z_2}{Z_1 + Z_2} \rightarrow \text{Find out}$$

Solving differential equation eliminated. Hence computationally simpler.

Complex Electric Permittivity

Youngs modulus

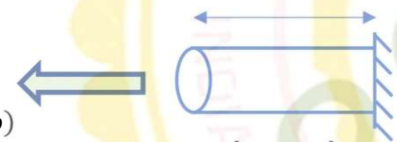
$$\bar{\gamma} = \gamma_{st}(1+\eta j)$$

$$\sigma = \sigma_o e^{j\omega t}$$

$$\epsilon = \frac{\sigma_o}{\gamma} e^{j(\omega t-\varphi)}$$

$$\sigma = \sigma_o e^{j\omega t}$$

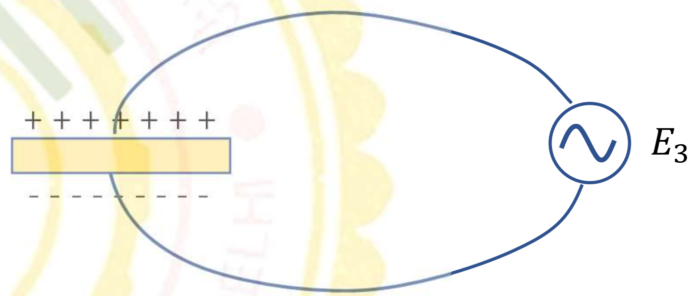
$$\epsilon = \frac{\sigma_o}{\gamma} e^{j(\omega t-\varphi)}$$



Electric permittivity

$$\overline{\epsilon_{33}^T} = \epsilon_{33}^T(1-\delta j)$$

Electric Loss tangent (Electrical damping)



Electrical changes lag behind the field

$$D_3 = \overline{\epsilon_{33}^T} E_3 + d_{31} T_1$$

Zero stress

$$D_3 = \epsilon_{33}^T (1-\delta j) \cdot E_o\, e^{j\omega t}$$

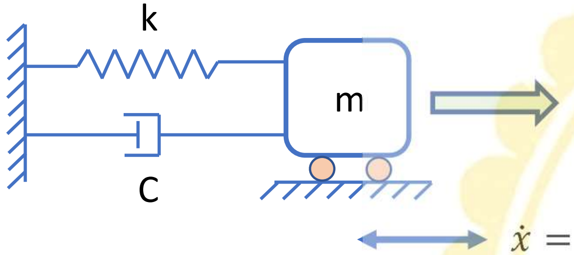
$$= \epsilon_{33}^T E_o \cdot e^{j(\omega t-\delta)}$$

$$= D_{3o} \cdot e^{j(\omega t-\delta)}$$

$e^{-j\delta}$ *for* $\delta \ll 1$

Analogy

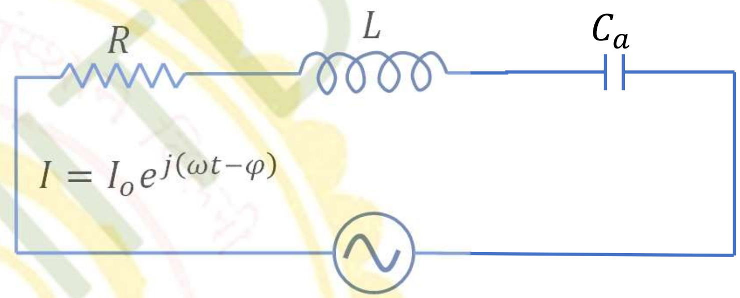
Mechanical System



$F = F_o e^{jwt}$

$\dot{x} = \dot{x}_o e^{j(\omega t - \varphi)}$

Electrical System



$I = I_o e^{j(\omega t - \varphi)}$

$V = V_o e^{jwt}$

Equivalent: $V \Leftrightarrow F$ $\quad \dot{x} \Leftrightarrow I$

$$Z = C + j\left(m\omega - \frac{k}{\omega}\right)$$

$$Z_e = R + j\left(L\omega - \frac{1}{C_a\omega}\right)$$

Equivalent: $C \Leftrightarrow R$ (Dissipative)

$m \Leftrightarrow L$

$k \Leftrightarrow C_a$ (Non- Dissipative)

If $(m, k) = 0$ or $(L, C_a) = 0 \implies$ No phase lag