DIRECT STIFFNESS METHOD FOR ANALYSIS OF SKELETAL STRUCTURES

http://web.iitd.ac.in/~sbhalla/cvl756.html

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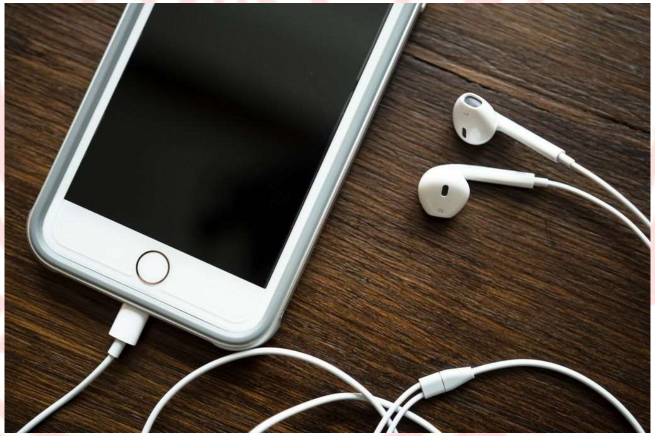
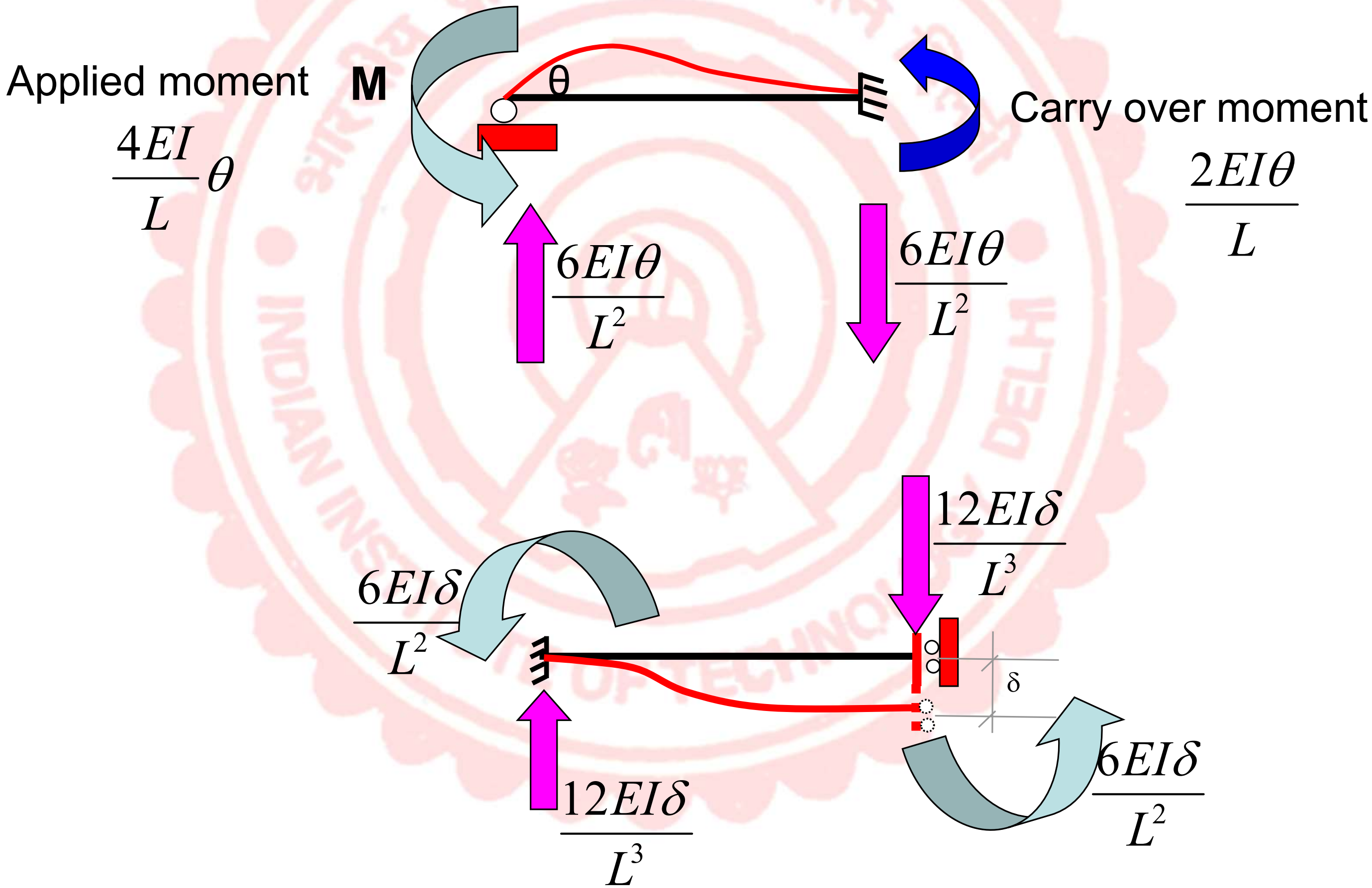
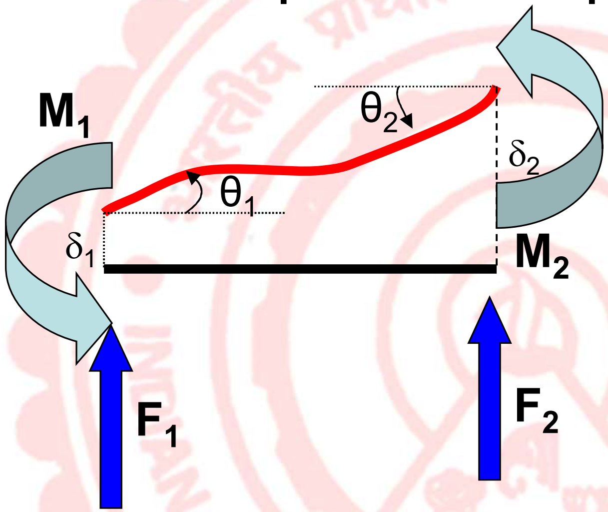


SLOPE DEFLECTION METHOD



Generalized slope deflection equations



$$M_1 = \frac{2EI}{L}\left[2\theta_1 + \theta_2 - \frac{3(\delta_2 - \delta_1)}{L}\right]$$

$$M_2 = \frac{2EI}{L}\left[\theta_1 + 2\theta_2 - \frac{3(\delta_2 - \delta_1)}{L}\right]$$

$$F_1 = -F_2 = \frac{M_1 + M_2}{L}$$

$$F_1 = \frac{2EI}{L^2}\left[3\theta_1 + 3\theta_2 - \frac{6(\delta_2 - \delta_1)}{L}\right] \qquad F_2 = -\frac{2EI}{L^2}\left[3\theta_1 + 3\theta_2 - \frac{6(\delta_2 - \delta_1)}{L}\right]$$

PUT IN MATRIX FORM

$$\begin{bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{bmatrix} = \begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{bmatrix} \delta_1 \\ \theta_1 \\ \delta_2 \\ \theta_2 \end{bmatrix}$$

- **Symmetric**
- **K_{ii} +ive**

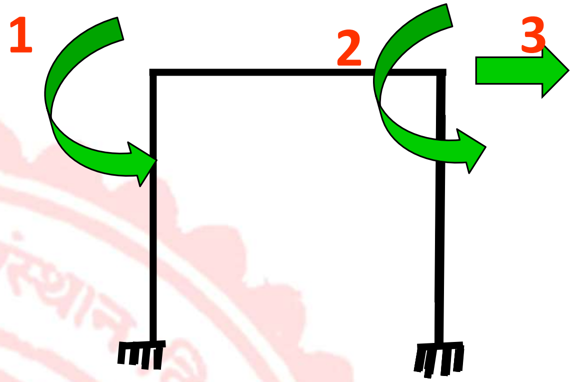
$$\{F\} = [K]\{\delta\}$$

Force vector

Stiffness matrix (symmetric)

Displacement vector

$$\begin{bmatrix} F_1 \\ F_2 \\ . \\ F_n \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} & . & K_{1n} \\ K_{21} & K_{22} & . & K_{2n} \\ . & . & . & . \\ K_{n1} & K_{n1} & . & K_{nn} \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ . \\ \delta_n \end{bmatrix}$$



n = Degrees of freedom (DKI)

F_i = Force along i^{th} degree of freedom

δ_i = Displacement along i^{th} degree of freedom

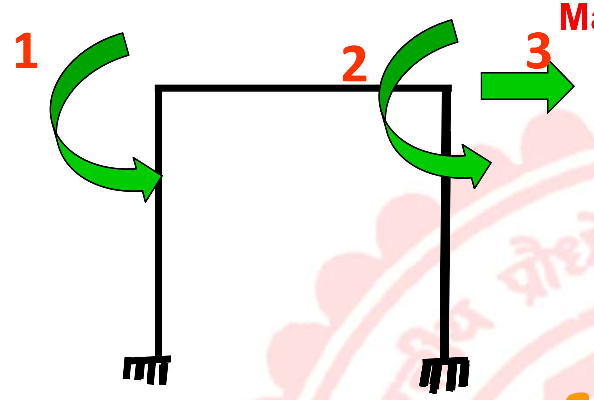
j^{th} col. of [K] :

Forces generated along the various degrees of freedom under a unit displacement along the j^{th} degree of freedom ($\delta_j = 1$), with all other degrees of freedom locked ($\delta_x = 0$, where $x \neq j$)

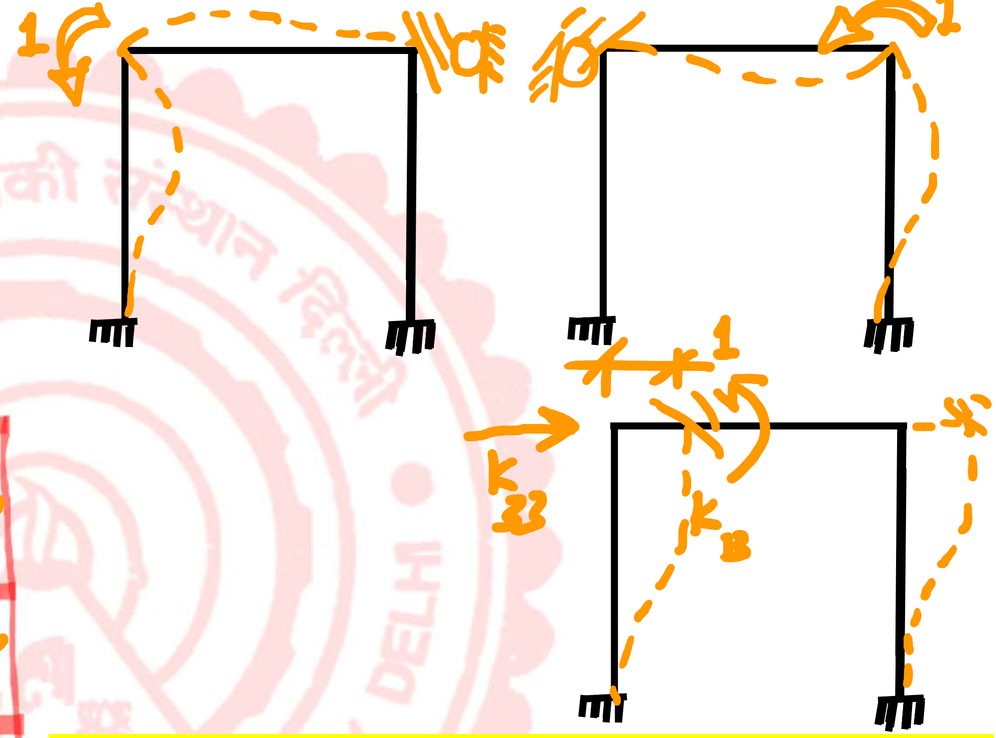
K_{ij} =Force generated along i^{th} degrees of freedom under a unit displacement along the j^{th} degree of freedom with all other degrees of freedom locked.

Matrix Stiffness Approach (MSA) :

The elements of [K] are obtained by first principles using the definition of k_{ij} from the deformation pattern of the structure and force-deformation relations of the members



Matrix Stiffness Approach (MSA) : Human judgment needed



	1	2	3
1	$\frac{8EI}{L}$		✓
2	$\frac{2EI}{L}$	$\frac{8EI}{L}$	✓
3	k_{13}	k_{23}	$\frac{24EI}{L^3}$

$\frac{6EI}{L^2}$

The direct stiffness approach (DSA), on the other hand, enables computation of the overall stiffness of any complicated structure using computer program, based on finite element formulation, without human visualization of the overall structure.

SIGNIFICANCE OF DIRECT STIFFNESS METHOD

ALL COMMERCIAL STRUCTURAL ENGINEERING ANALYSIS PACKAGES ARE BASED ON THE DIRECT STIFFNESS APPROACH.

UNDERSTANDING AND IMPLEMENTING THE CONCEPTS WILL HELP YOU IN:

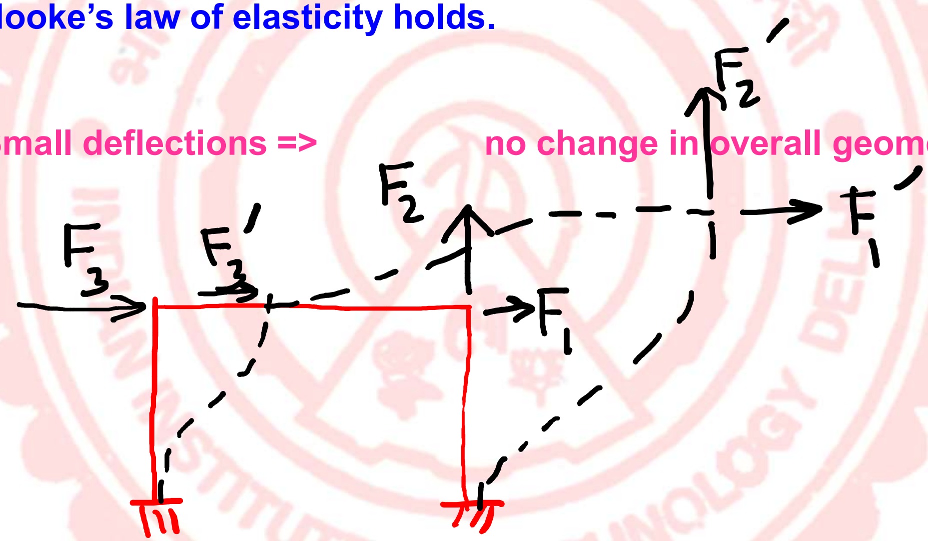
1. **MAKING YOUR OWN CUSTOMIZED RESULT ORIENTED SOFTWARE WITHOUT SPENDING ANY PENNY.**
2. **USING THE EXISTING SOFTWARE IN ERROR FREE MANNER, WITH UNDERSTANDING, RATHER THAN AS A "BLACK BOX" APPROACH.**

DIRECT STIFFNESS METHOD: ASSUMPTIONS

1. Restricted to frame and truss structures (skeletal structures) only. Members assumed as line elements (passing through neutral axis) with lumped sectional properties. At first, we restrict analysis to prismatic members only.

2. Hooke's law of elasticity holds.

3. Small deflections => no change in overall geometry of structure.



Geometric non-linearity introduced.

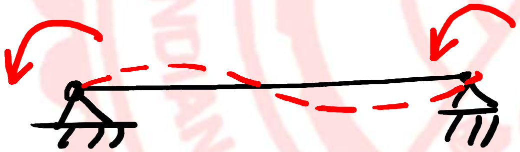
DIRECT STIFFNESS METHOD: ASSUMPTIONS

4. **Plane sections remain plane after bending.**

5. **In bending mode, very small slope**

$$\text{Curvature} = \frac{d^2y/dx^2}{\left[1+(dy/dx)^2\right]^{3/2}} \approx \frac{d^2y}{dx^2}$$

6. **If displacement takes place normal to member, no change in length of the member. Change in length of an element due to flexural deformation (curvature effects) is also negligible.**



Axial & bending effects independent

DIRECT STIFFNESS METHOD: ASSUMPTIONS

7. Principle of superposition holds good.
   - Loads can be superimposed
   - Boundary conditions can be superimposed
   - Displacements can be superimposed
   - BMD, SFD can be superimposed.

IMPORTANT:

All assumptions of slope deflection method are repeated except one…..

We have discarded the assumption regarding inextensibility of the members……

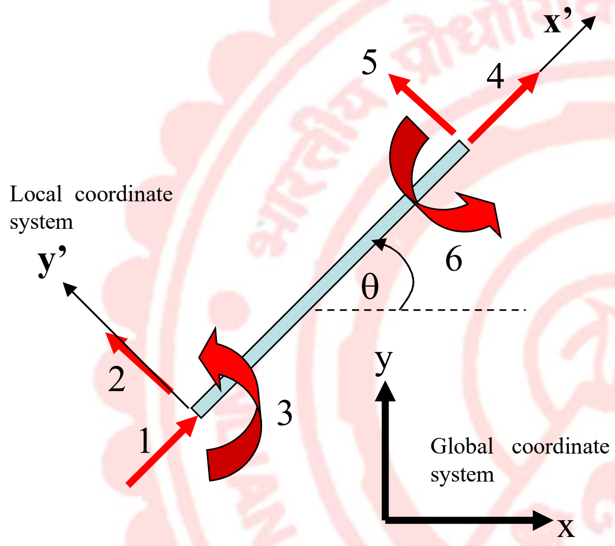
Unlike manual approach, digital computers will not have no problem in tackling additional degrees of freedom.

DIRECT STIFFNESS METHOD FOR COMPUTER APPLICATIONS

- Each individual member is treated as structure (called element).
- Stiffness matrix of each individual element is obtained.
- Total stiffness matrix of the entire structure is then computationally obtained by superimposing the matrices of elements, without human intervention.
- Hence, analysis can be broken down into small steps and programmed, in a **finite element procedure**.

GENERATION OF ELEMENT STIFFNESS MATRIX

2D STRUCTURES



$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{Bmatrix} = \begin{bmatrix} \frac{EA}{L} & & & & & \\ 0 & \frac{12EI}{L^3} & & & & \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & & & \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & & \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{Bmatrix}$$

z axis normal to plane of board towards viewer

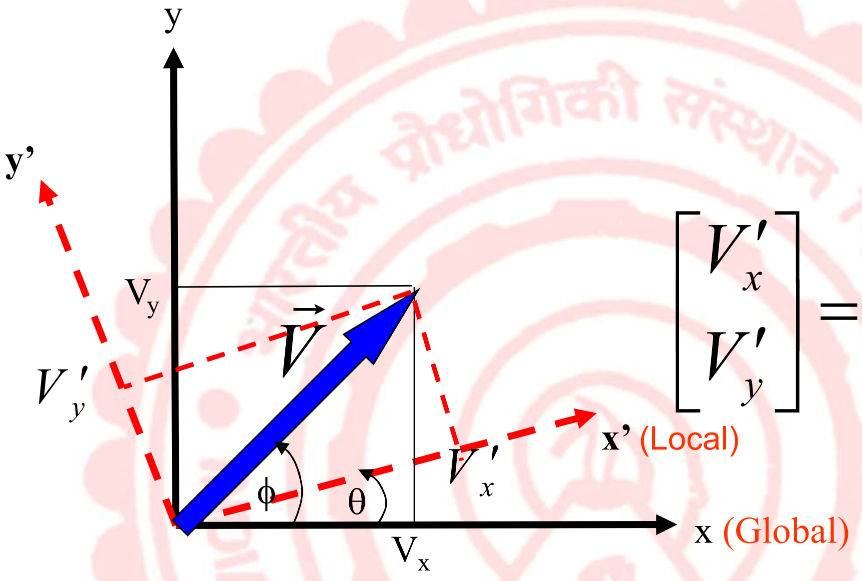
In short form, ***{f}= [k]$_L$ {d}***

[k]$_L$ = Element stiffness matrix with respect to **local coordinate system**.

TRANSFORMATION OF COORDINATE SYSTEM

- At a joint, members of different orientations may meet.
- The forces and displacements at member ends cannot be easily related.
- To consider equilibrium of the joint and compatibility of member displacements, the member end forces and displacements must be transformed to a common coordinate system.

$\vec{V}$ = A vector (force or displacement)



$$\begin{bmatrix} V'_x \\ V'_y \end{bmatrix} = \begin{array}{cc} l_1 & m_1 \\ \end{array} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} V_x \\ V_y \end{bmatrix}$$

$$l_2 \quad m_2$$

l_1	m_1
l_2	m_2

Direction cosines

$\vec{V} = V_x \hat{i} + V_y \hat{j}$

$= (V\cos\varphi)\hat{i} + (V\sin\varphi)\hat{j}$

$\vec{V} = V_x' \hat{i}' + V_y' \hat{j}'$

$= V\cos(\varphi-\theta)\, \hat{i}' + V\sin(\varphi-\theta)\, \hat{j}'$

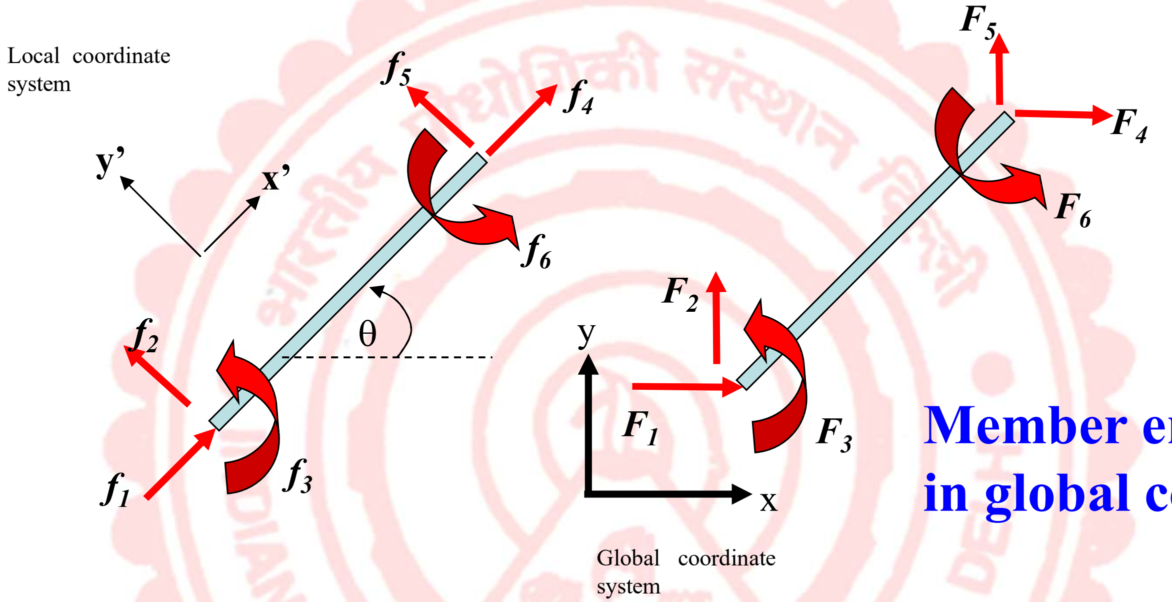
$= [V\cos\varphi\cos\theta + V\sin\varphi\sin\theta]\, \hat{i}' + [V\sin\varphi\cos\theta - V\cos\varphi\sin\theta]\, \hat{j}'$

$= [V_x\cos\theta + V_y\sin\theta]\, \hat{i}' + [-V_x\sin\theta + V_y\cos\theta]\, \hat{j}'$

Hence

$$V_x' = V_x\cos\theta + V_y\sin\theta$$
$$V_y' = -V_x\sin\theta + V_y\cos\theta$$

MEMBER FORCES IN GLOBAL & LOCAL COORDINATES

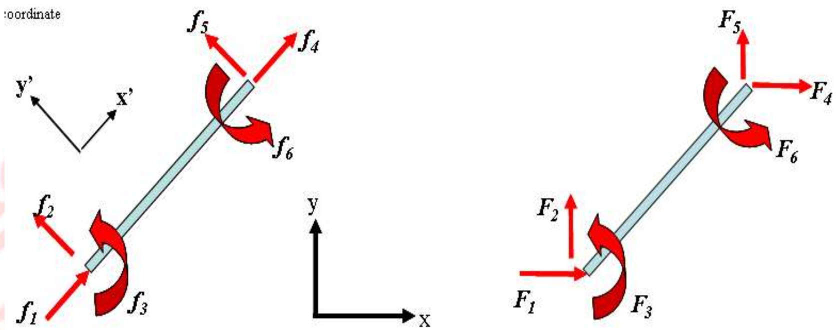


Member end forces in global coordinates

Member end forces in local coordinates

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$$

$$\begin{bmatrix} f_4 \\ f_5 \\ f_6 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_4 \\ F_5 \\ F_6 \end{bmatrix}$$



$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \boxed{R} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} \qquad \mathbf{1}$$

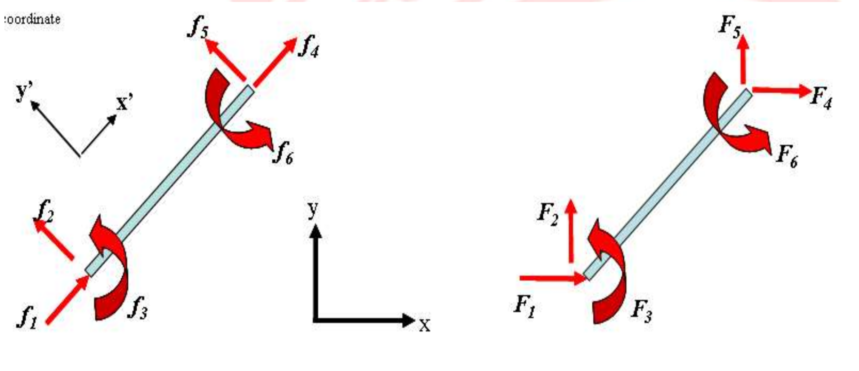
$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \boxed{R^{-1}} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \boxed{R^{T}} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

$R^{-1} = R^{T}$ **R is an orthogonal matrix.**

R = JOINT TRANSFORMATION MATRIX

Similarly
$$\begin{bmatrix} f_4 \\ f_5 \\ f_6 \end{bmatrix} = \boxed{R} \begin{bmatrix} F_4 \\ F_5 \\ F_6 \end{bmatrix} \qquad \mathbf{2}$$

Combining equations **1** and **2**



$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{bmatrix} = \begin{bmatrix} R & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & R \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{bmatrix}$$

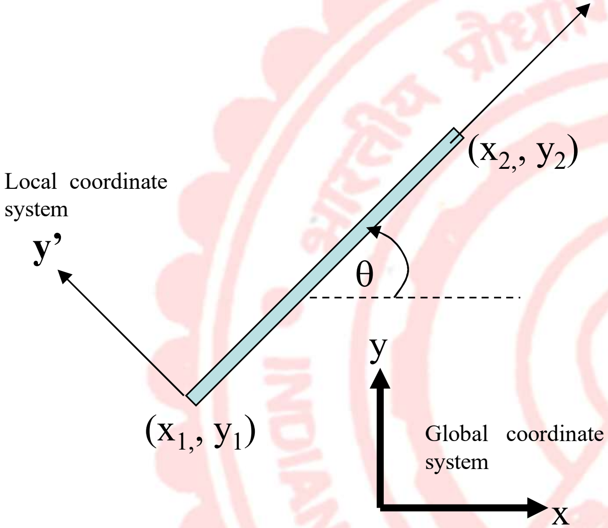
$$\{ f \} = [T] \{F\} \qquad T = \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}$$

[T] = MEMBER TRANSFORMATION MATRIX

Similarly for displacements $\{ d \} = [T] \{D\}$

HOW WILL PROGRAM OBTAIN THE NECCESARY INFORMATION FOR COORDINATE TRANSFORMATION??

User should provide the coordinates of all joints….



$$\cos\theta = \frac{x_2 - x_1}{L}$$

$$\sin\theta = \frac{y_2 - y_1}{L}$$

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$R = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T] = \begin{array}{|c|c|} \hline R & 0 \\ \hline 0 & R \\ \hline \end{array} \qquad T^{-1} = T^{T}$$

HOW IS TRANSFORMATION UTILIZED??

$$\{f\} = [k]_L \{d\}$$

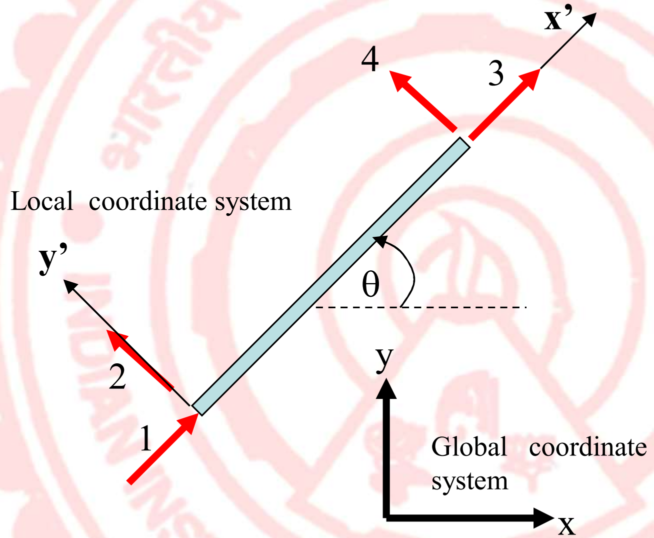
$$[T]\{F\} = [k]_L [T]\{D\}$$

$$\{F\} = [T]^T [k]_L [T]\{D\}$$

$[K]_G$ Stiffness matrix of member in global coordinates

$$[K]_G = [T]^T [k]_L [T]$$

SPECIAL CASE: TRUSS STRUCTURES

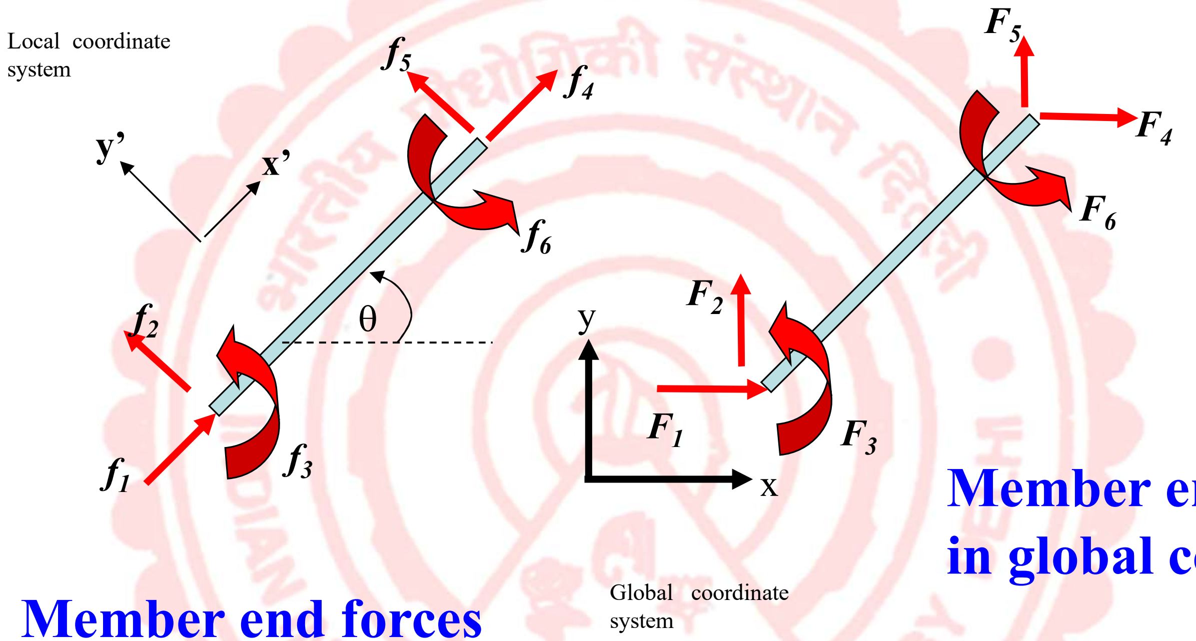


$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{Bmatrix} = \begin{bmatrix} \frac{EA}{L} & 0 & -\frac{EA}{L} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{EA}{L} & 0 & \frac{EA}{L} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix}$$

$$R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

No member end moments or rotations

MEMBER FORCES IN GLOBAL & LOCAL COORDINATES



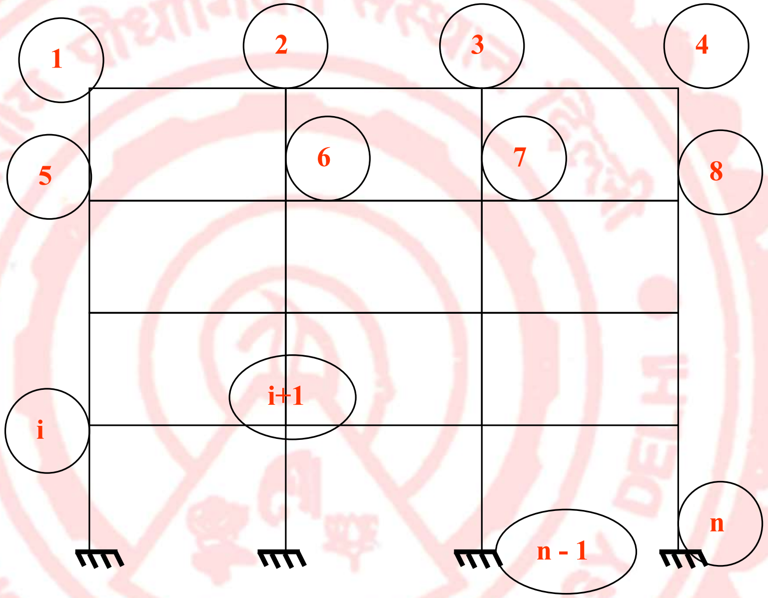
Member end forces in global coordinates

Member end forces in local coordinates

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$$

$$\begin{bmatrix} f_4 \\ f_5 \\ f_6 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_4 \\ F_5 \\ F_6 \end{bmatrix}$$

GENERATION OF TOTAL STRUCTURAL STIFFNESS MATRIX



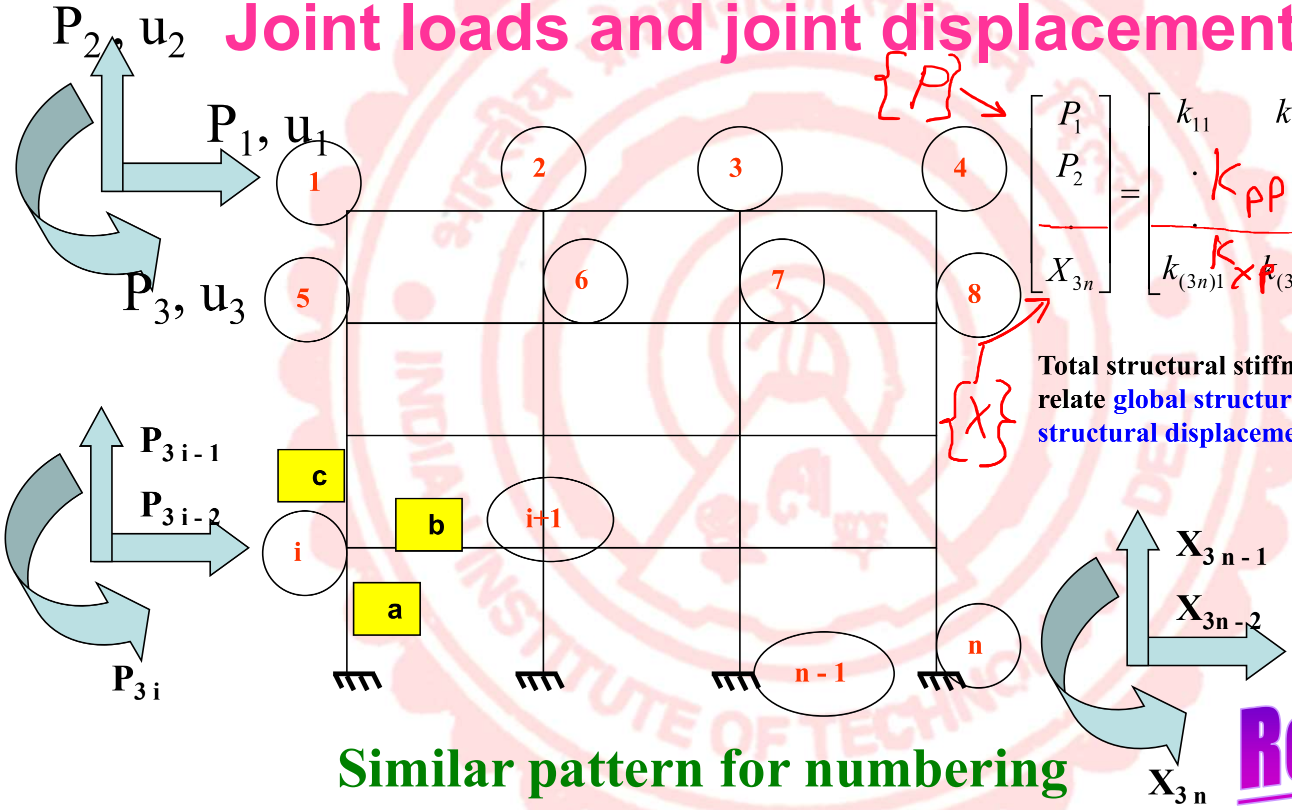
We shall first derive formulations for simple 2D case:
(1)Supports are fixed (2) All joints are rigid with no internal hinges.
(3) Joints can be sequentially numbered as above

We shall introduce complications into analysis one by one.

NUMBERING SCHEME

Joints numbered sequentially, restrained joints numbered in end

Joint loads and joint displacements



{P} {u_P}

$$\begin{bmatrix} P_1 \\ P_2 \\ \cdot \\ X_{3n} \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & \cdot & k_{1(3n)} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ k_{(3n)1} & k_{(3n)2} & \cdot & k_{(3n)(3n)} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ u_{3n} \end{bmatrix}$$

K_{PP} K_{Px} K_{xP} K_{xx}

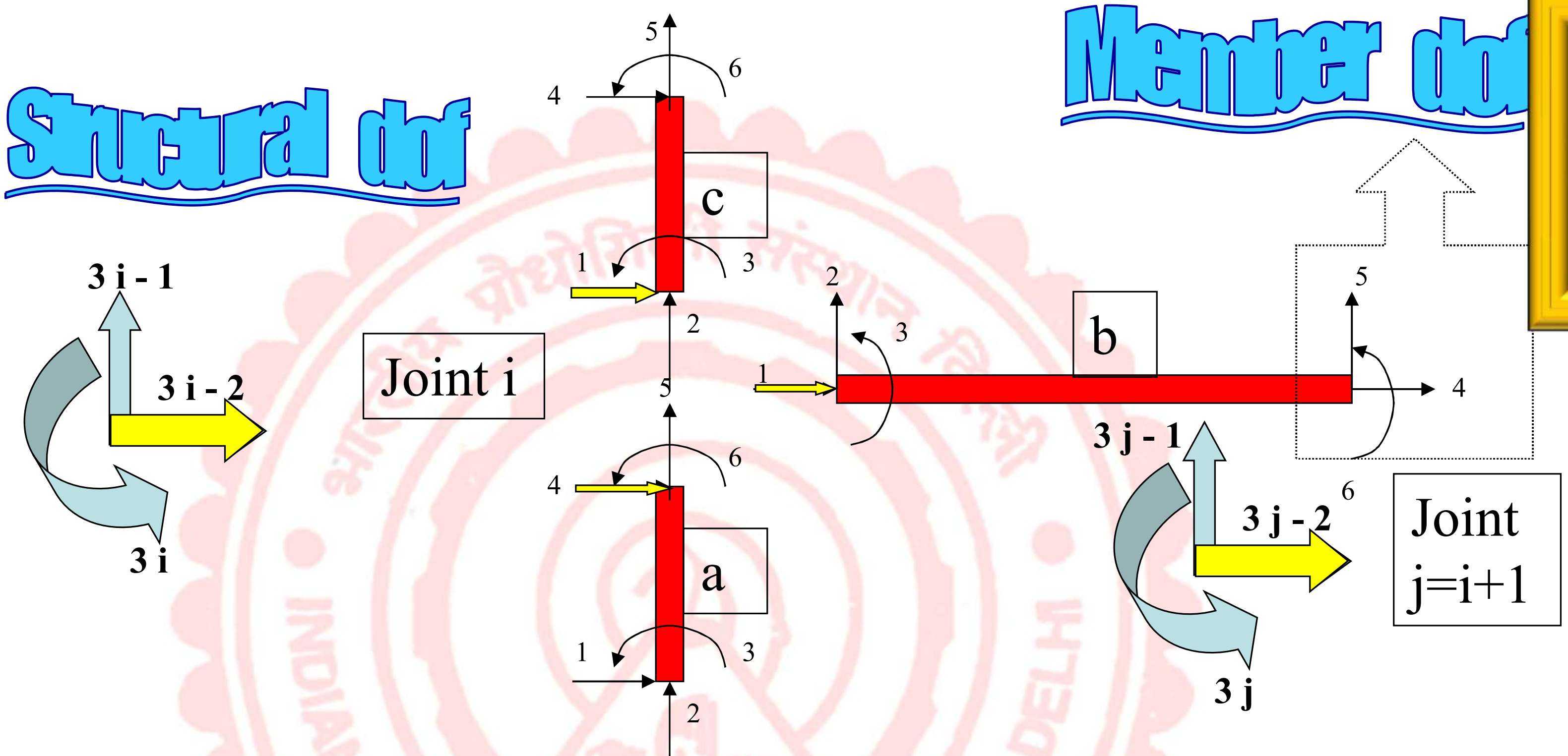
{X} {u_x} u_{x3n}

Total structural stiffness matrix should relate global structural loads to global structural displacements

Reactions

Similar pattern for numbering of displacements

Degrees of freedom shall include those at supports also



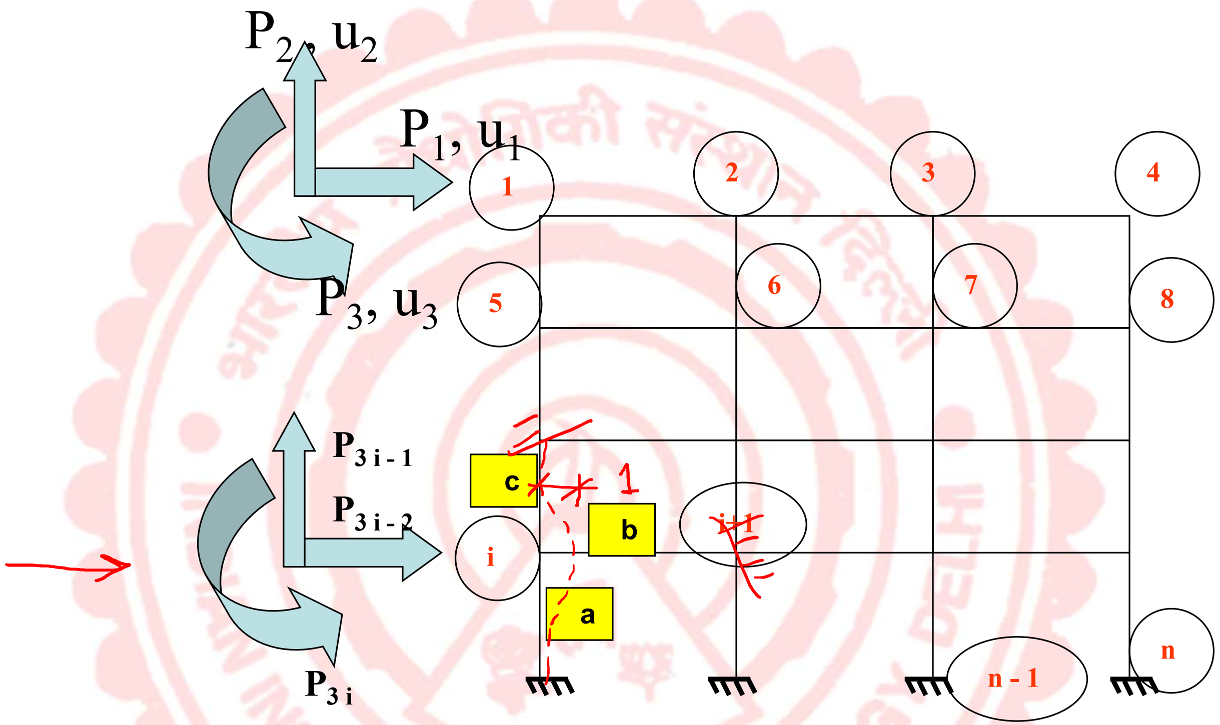
Member degrees of freedom: From element point of view (1..6)

Structural degrees of freedom: From global (overall structures) point of view (1..3n)

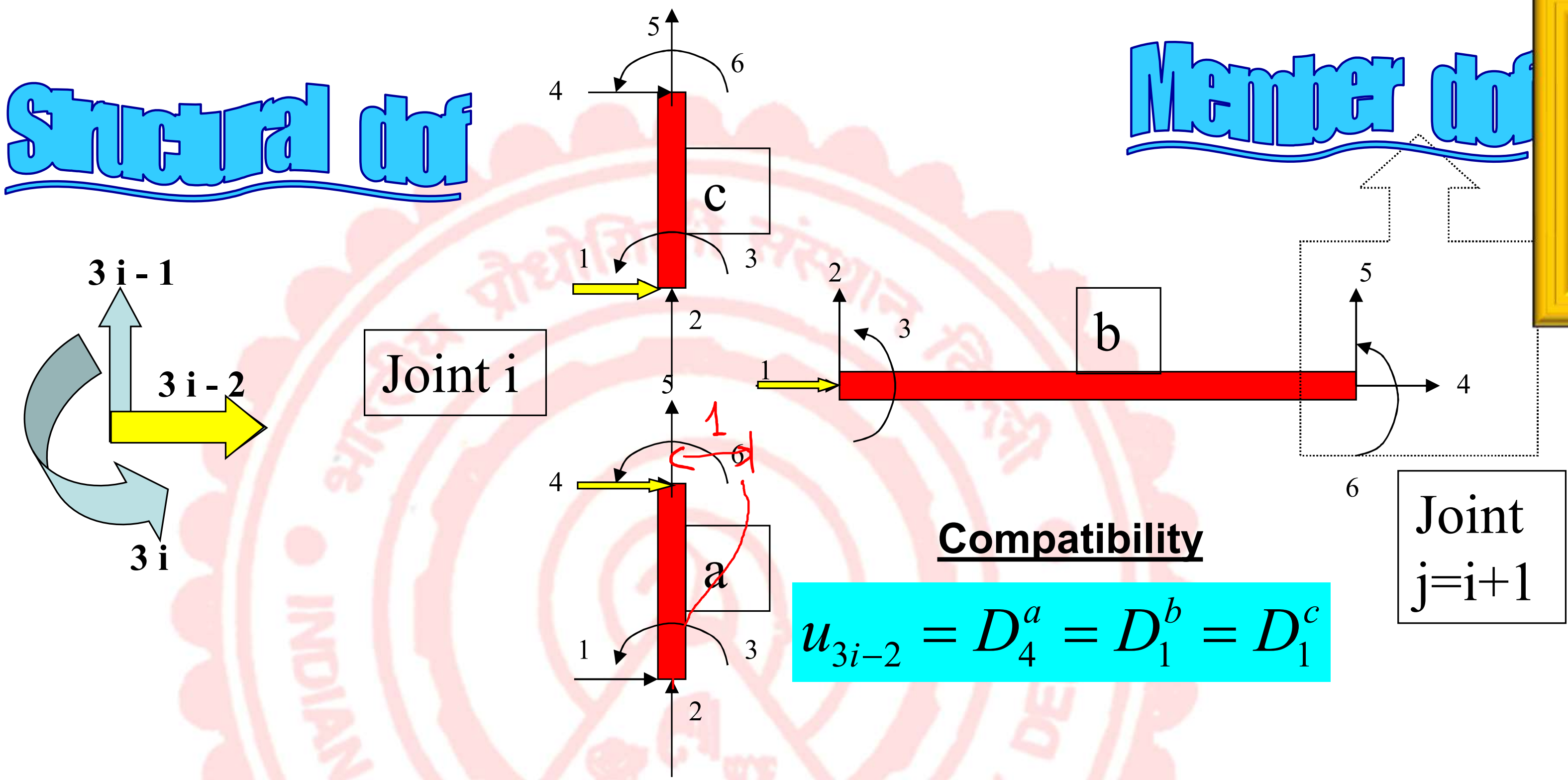
Each member degree of freedom in global coordinates (1,2,…,6) corresponds to a particular structural degree of freedom (1,2,….,3n).

COMPATIBILITY $u_{3i-2} = D_4^a = D_1^b = D_1^c$

EQUILIBRIUM CONDITIONS



Let a unit displacement be applied along d.o.f (3i-2) and all other d.o.f. =0



Compatibility

$$u_{3i-2} = D_4^a = D_1^b = D_1^c$$

Let a unit displacement be applied along d.o.f (3i-2) and all other d.o.f. =0

By joint equilibrium,

P_{3i-2} = Sum of member end forces of a, b, c in X-direction

$P_{3i-2} = F^a_4 + F_1^b + F^c_1$

Recall: k_{mn} = Force induced along d.o.f.'m' due to unit displacement along d.o.f 'n', all other displacements maintained zero.

$$K_{3i-2,\ 3i-2} = k^a_{44} + k^b_{11} + k^c_{11}$$

$$K_{3i-2,\ 3i-2} \quad = \quad k^a_{44} + k^b_{11} + k^c_{11}$$

Element of total structural stiffness matrix

Elements of member stiffness matrices in global coordinates

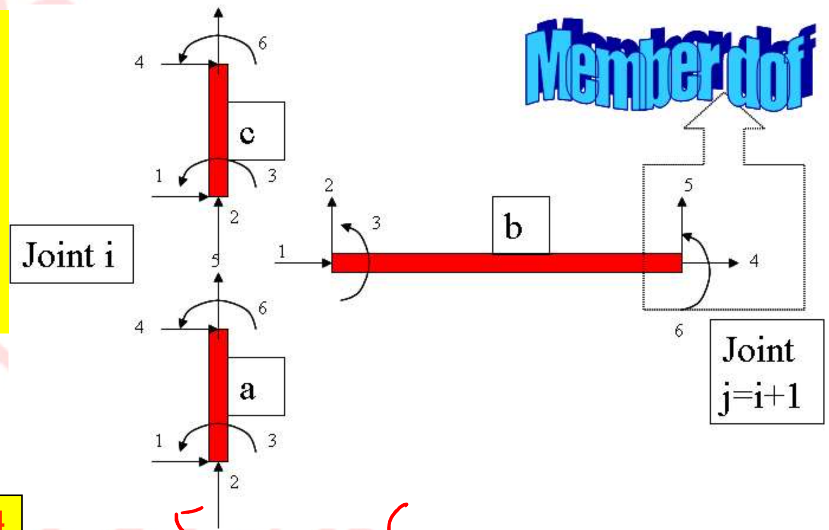
- An element of $[K]_{TS}$ can be obtained by summing the elements of member stiffness matrices (in global coordinates) of corresponding d.o.f from members that frame into that joint.

- In order to carryout smoothly, we follow **Code Number Approach.**

Each member degree of freedom in global coordinates (1,2,…,6) corresponds to a particular structural degree of freedom (1,2,….,3n). This information can be stored in the association matrix of the member.

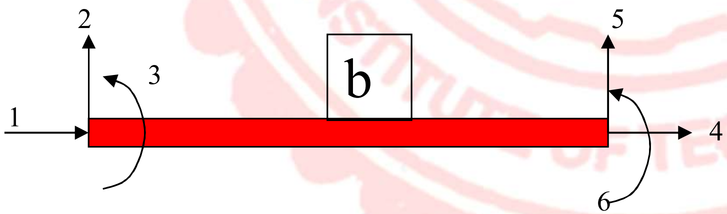
K_{34} of $[K]_L$ will correspond to $K_{3i,\,3(i+1)-2}$ of $[K]_{TS}$

(i.e. will transfer to that location)

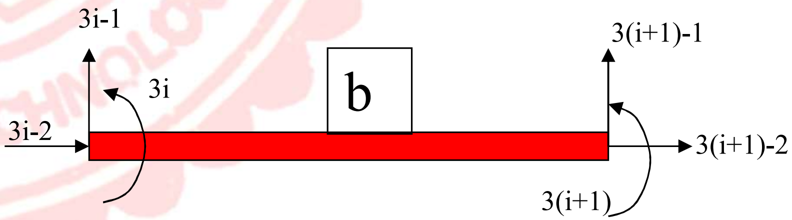


For member ‘b’, the association matrix is

1	2	3	4	5	6
3i-2	*3i-1*	*3i*	*3(i+1)-2*	*3(i+1)-1*	*3(i+1)*



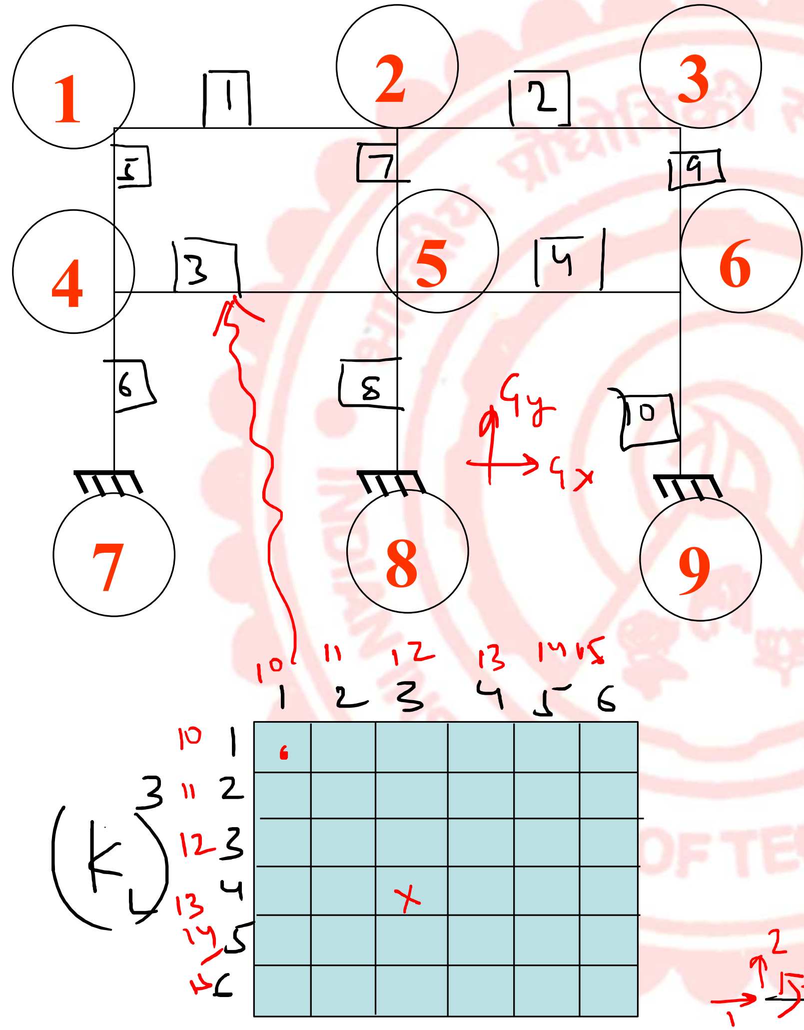
Member dof

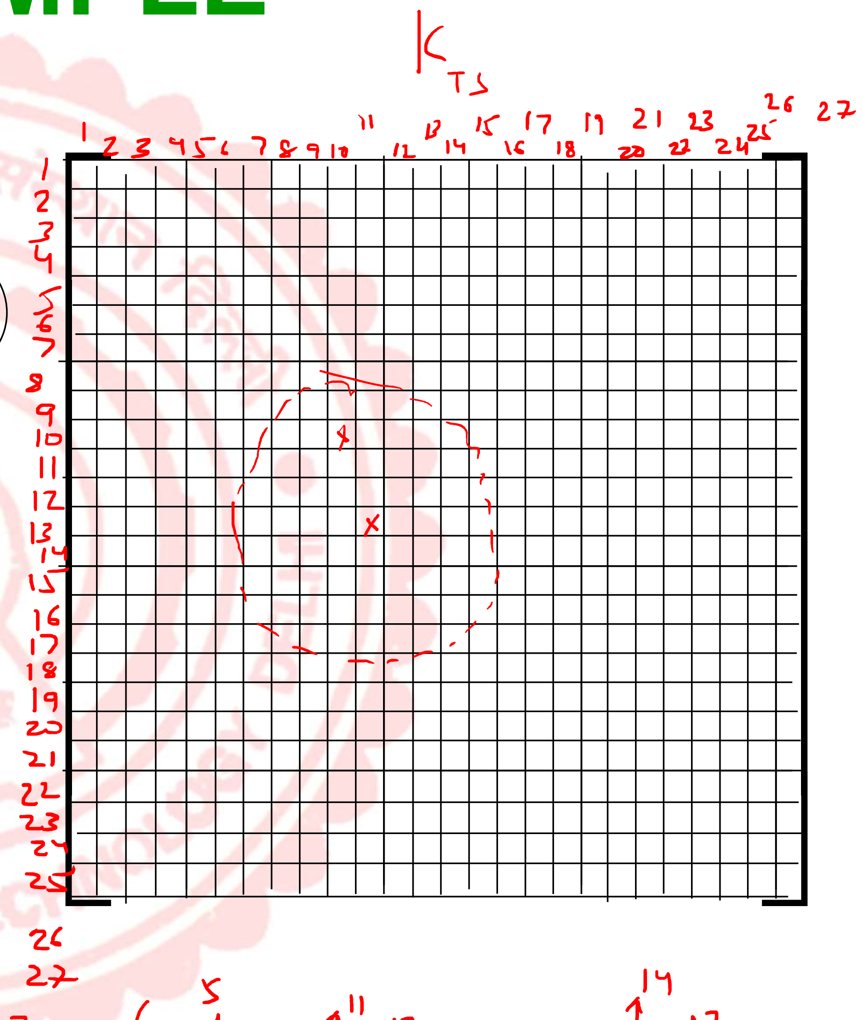


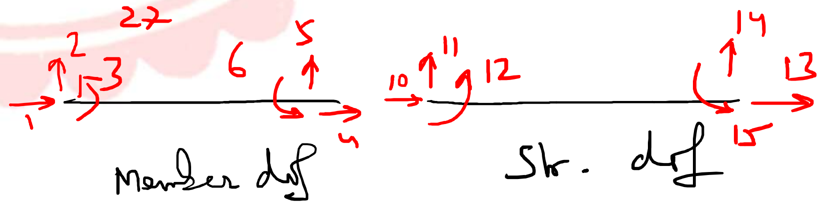
Structural dof

$A[3] = \boxed{10 \mid 11 \mid 12 \mid 13 \mid 14 \mid 15}$

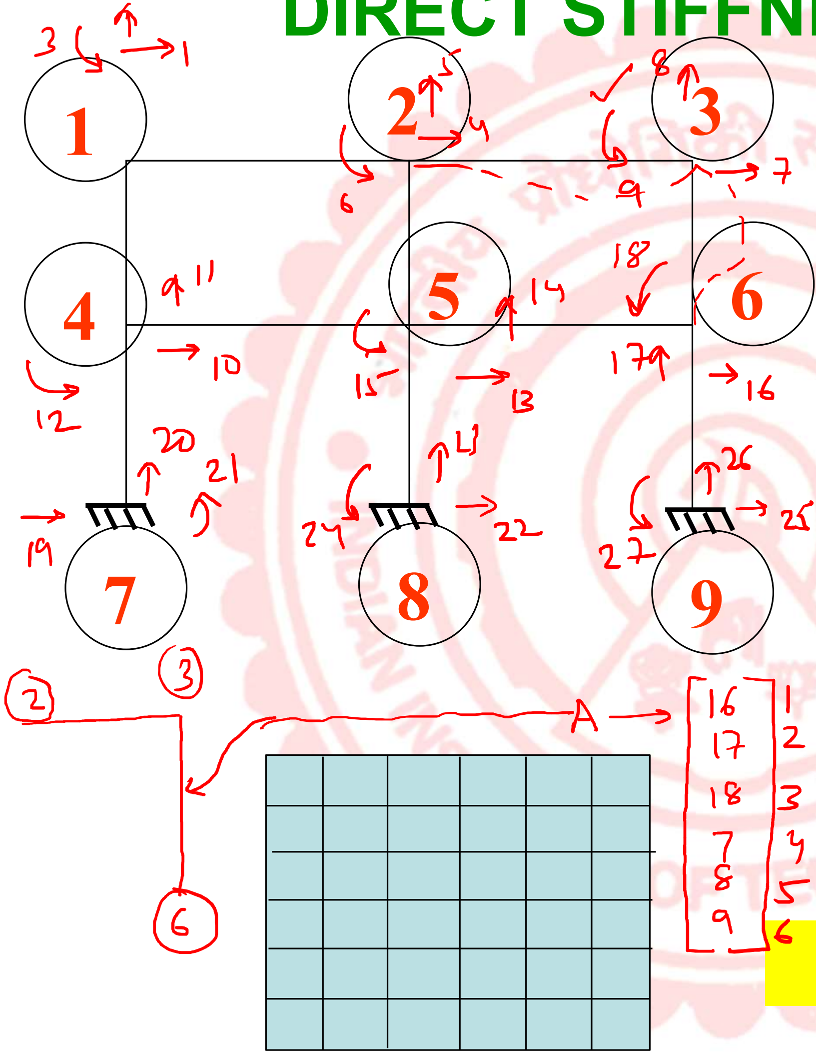
EXAMPLE

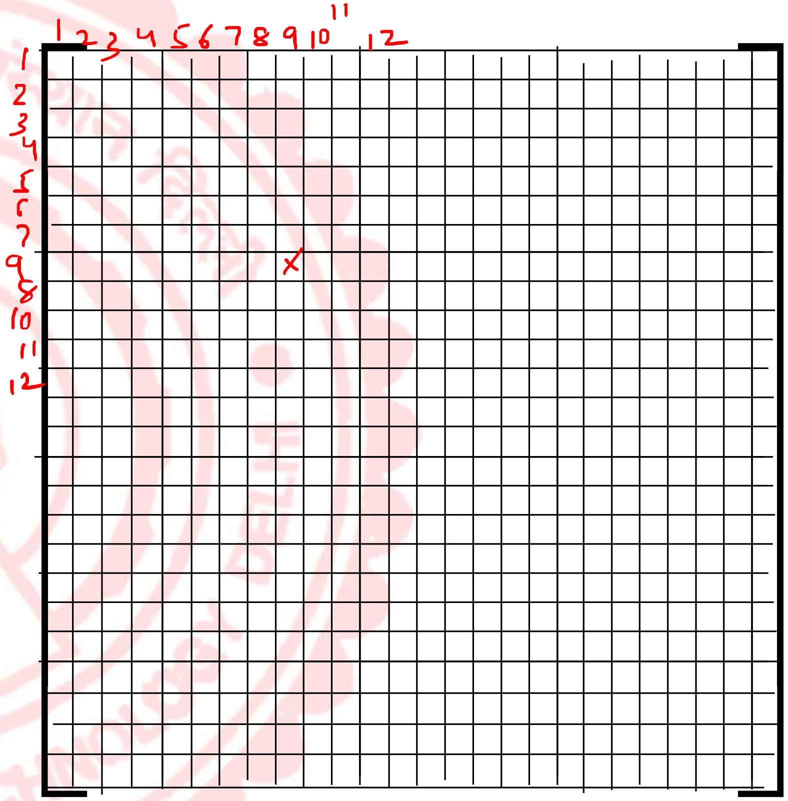






K_{TS} GENERATION: MATRIX STIFFNESS VS DIRECT STIFFNESS APPROACH

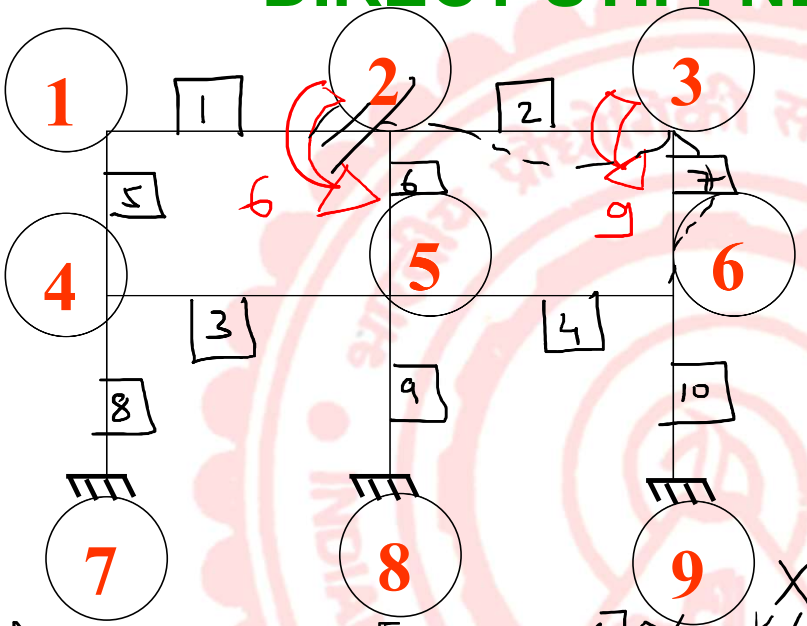


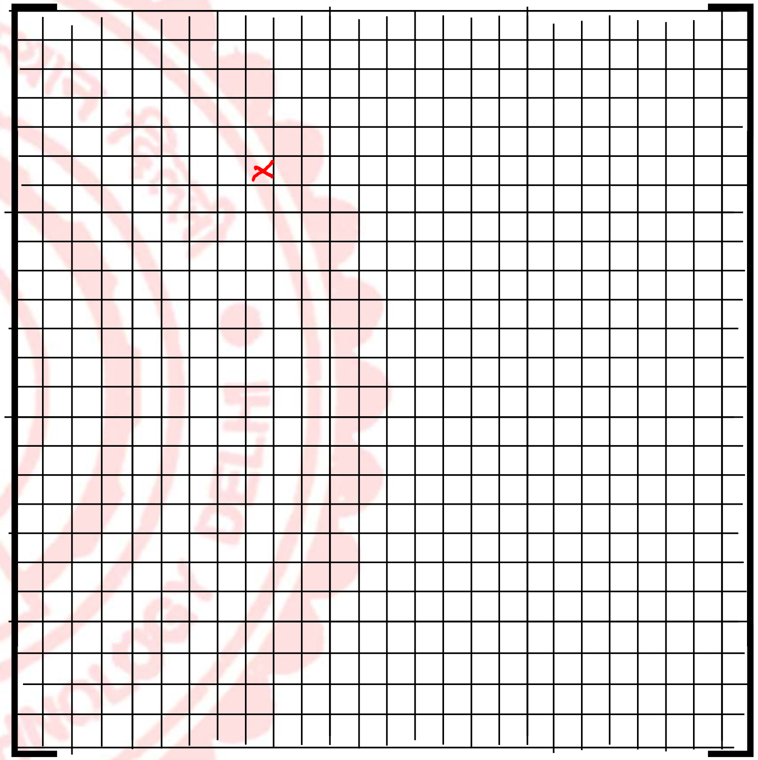


K_{TS} (9,9) =??

$= K^{9}_{66}(\text{mem } 6\text{-}3) + K_{66}(\text{mem } 2\text{-}3)$

K_{TS} GENERATION: MATRIX STIFFNESS VS DIRECT STIFFNESS APPROACH

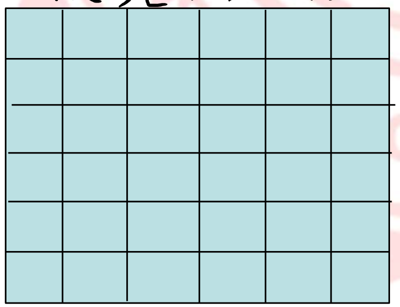




Mem
1
2
6
7

K_{69}

A(1)[1,2,3,4,5,6]
A(2)[4,5,6,7,8,9]



A^6[13,14,15,4,5,6]

A_7[16,17,18,7,8,9]

K_{TS} (6,9) =??

HOW TO GENERATE THE TOTAL STRUCTURAL STIFFNESS MATRIX

- All joints of the structure should be numbered sequentially, starting from the unstrained joints.
- Restrained joints should be numbered in the end.
- Initialize the total structural stiffness matrix to '0'.
- Consider each member; compute its member stiffness matrix in global coordinates.
- Then send its elements into appropriate location of the global stiffness matrix of the entire structure, one at a time.
- Repeat this process for each member; keep adding its elements to the appropriate elements of the total structural stiffness matrix.
- Finally, the total stiffness matrix of the structure will result.

STEPWISE PROCEDURE FOR PROGRAMMING

1.Label all elements (or members) 1…..m.

2.Label all joints 1….n, first unrestrained, then the restrained ones. D.O.F associated with i^{th} node: $3i-2$, $3i-1$, $3i$. Hence, all d.o.f are also numbered.

3. Compute the size of structural stiffness matrix & initialize it to 0.

STEPWISE PROCEDURE FOR PROGRAMMING (Contd…)

4. Repeat for each element (from i=1 to m)

- Compute $[k]_L$
- Compute $[R]$ from end coordinates.
- Compute $[k]_G = [T]^T[k]_L[T]$
- Establish association matrix (from node number of the two nodes of member.
- Transfer each element of $[k_G]$ to appropriate location of $[K]_{TS}$

  $(k_{TS})_{ij} = \Sigma(k_G)_{mn}$

  Extends over all members meeting at a joint.

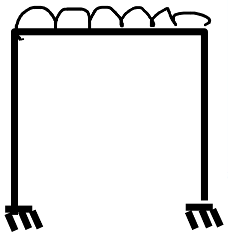
  m: Corresponds to i^{th} dof and n to the j^{th} dof.

Need to do this process 36 times for each member, no discount from symmetry….Why?

STEPWISE PROCEDURE FOR PROGRAMMING (Contd…)

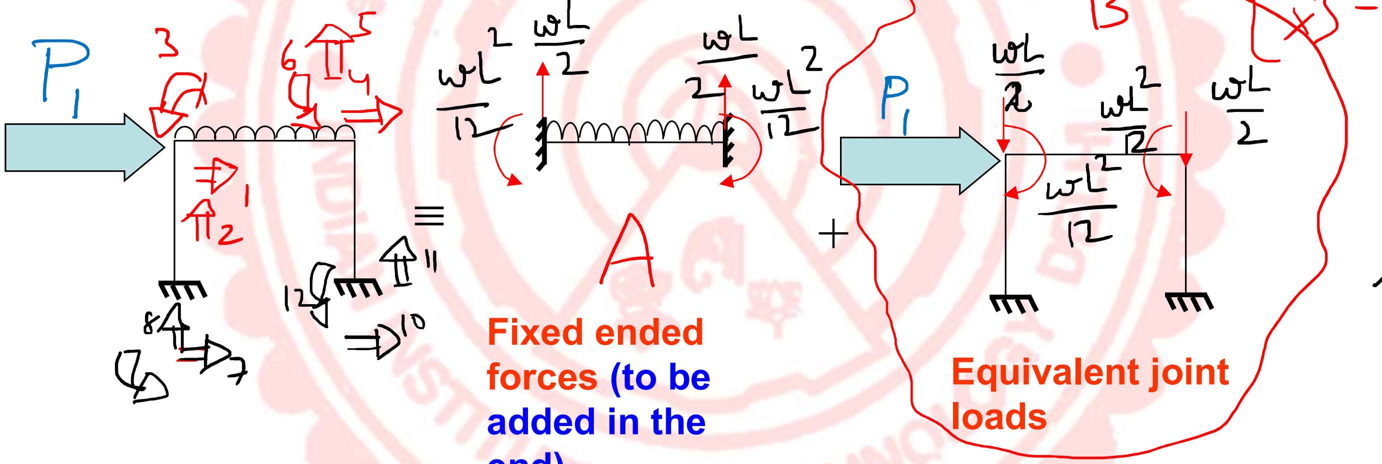
$K_{TS} = \boxed{\begin{matrix} 6\times 6 & \\ & 6\times 6 \end{matrix}}_{12\times 12}$

To be analysed



5. Obtain nodal loads P

- Direct nodal loads
- Equivalent nodal loads (from member loads such as distributed loads)



$\begin{Bmatrix} P \\ X \end{Bmatrix} =$

P_1	1
$-wL/2$	2
$-wL^2/12$	3
0	4
$-wL/2$	5
$+wL^2/12$	6
X_7	
X_8	
X_9	
X_{10}	
X_{11}	
X_{12}	

Equivalent joint load contributed by a member = $-[T]^T \{f\}^f$

(Global coordinates)

STEPWISE PROCEDURE FOR PROGRAMMING (Contd…)

6. Set up equations

known known

$$\begin{bmatrix} P \\ X \end{bmatrix} = \begin{bmatrix} K_{PP} & K_{PX} \\ K_{XP} & K_{XX} \end{bmatrix} \begin{bmatrix} u_P \\ u_X \end{bmatrix}$$

? (X) ? (u_P)

0, known value (u_X)

u_x : prescribed displacements : corresponding to X (unknown reactions)

u_p : un-prescribed displacements: corresponding to P (known loads)

Let $\{u_x\} = 0$ (no support movement)

$\{P\} = [k_{pp}] \{u_p\}$

$\therefore \{u_p\} = [k_{pp}]^{-1} \{P\}$ and hence $\{X\} = [k_{xp}] \{u_p\}$

If not zero

$\{P\} = [k_{pp}] \{u_p\} + [k_{px}] \{u_x\}$

$\{u_p\} = [k_{pp}]^{-1} \boxed{\{P\} - [k_{px}] \{u_x\}}$

Support settlement

Reactions $\{X\} = [k_{xp}]\{u_p\} + [k_{xx}]\{u_x\}$

STEPWISE PROCEDURE FOR PROGRAMMING (Contd…)

7. Member end forces (go back to member level)

For each member (from 1 to m), do-

a) Using association matrix, get nodal displacements $\{D\}$ (global coordinates)

Eg, for this member

$D_1=u_{13}$

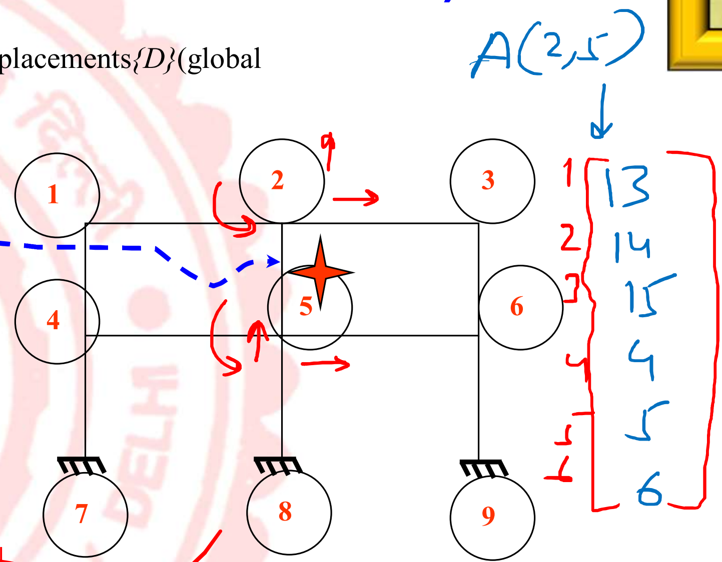
$D_2=u_{14}$

$D_3=u_{15}$

$D_4=u_4$

$D_5=u_5$

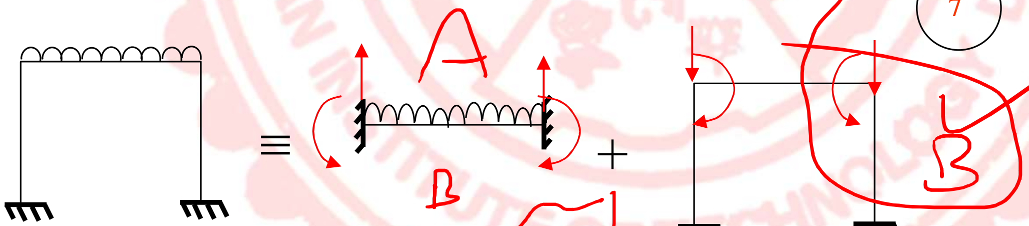
$D_6=u_6$



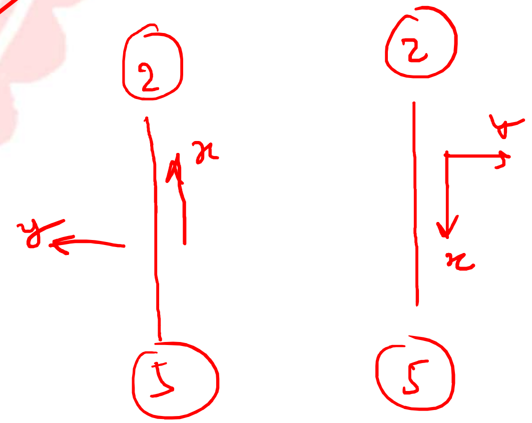
(b) $[d]=[T][D]$

Member end forces → (c) $\{f\}=[k]_L\{d\}$

(d) Correction for fixed ended action



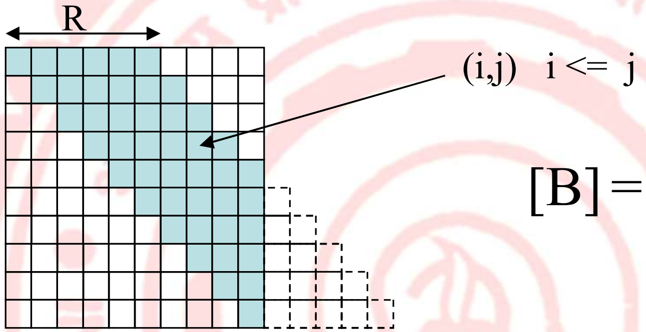
$\{f\}=[k^L][d]+[f]^f$ A

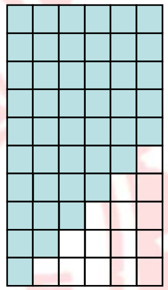


EFFICIENT STORAGE SCHEME FOR K_{PP}

$[K_{pp}]$ is a huge matrix. For example, for n= 100 unrestrained joints, it will be 300x300 in size. i.e. $9x10^4$ elements.

However, the fact is that it is symmetric and banded (why??)

$[K]_{pp} =$ 

$[B] =$ 

All elements outside the band are zero

Therefore, we only need to generate elements which are within the band. Generally, this is achieved by storing right half band in a rotated rectangular matrix.

An element (i,j) in the original matrix will go to:

Row = i

Column = c=(j-i+1)   in the banded matrix

Diagonal element k_{ii} : k_{i1}

Correlation between half banded and full [K]

Element $(i, r) \longrightarrow (i, c+i-1)$

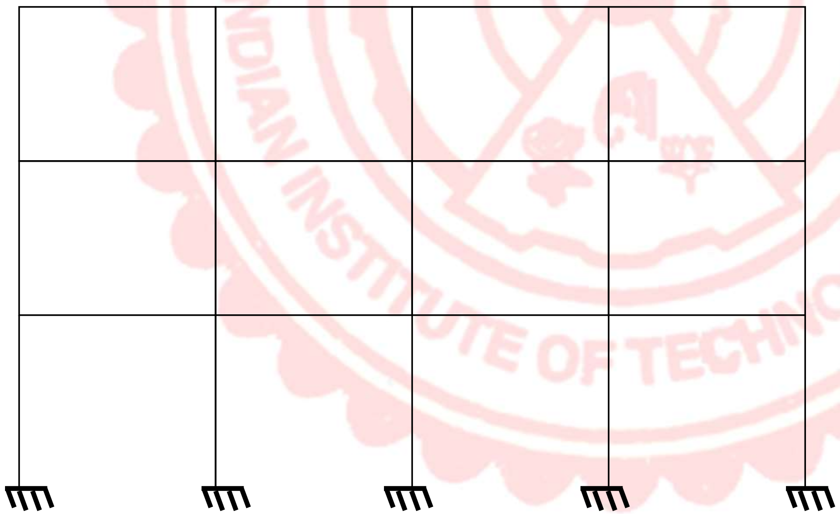
Diagonal $(i, 1) \longrightarrow (i, i)$

HOW TO COMPUTE 'R' , THE HALF BAND WIDTH

- Depends on structure size and also how we do numbering of joints
- For each member find –

  x = (max d.o.f. – min d.o.f) +1
- The max value of x will be equal to "R", the half band width. We need to store $[k]_{pp}$, $[k]_{px}$, $[k]_{xx}$

BASIS OF THIS FORMULATION ??



Horizontal numbering

$[k]_{pp}$ = 45x45

Half band width = 18

Vertical numbering

$[k]_{pp}$ = 45x45

Half band width = 12

CONCLUSION??

INVERSE OF $[K]_{PP}$: CHOLESKY'S ALGORITHM

To solve –

[P] = [k

$$[P] = [k_{pp}]\,[u_p] \qquad\qquad (\text{if } [u_x]=0)$$

$$\text{or } [P^*] = [k_{pp}][u_p] \qquad\qquad (\text{if } [u_x] \text{ is not equal to } 0)$$

$K\,u = P$ {let us relax notation}

Since K is symmetric

$K = V^T\,V$

Where V is upper triangular matrix.

$$\begin{bmatrix} K_1 & K_2 & K_3 & K_4 \\ K_5 & K_6 & K_7 & K_8 \\ K_9 & K_{10} & K_{11} & K_{12} \\ K_{13} & K_{14} & K_{15} & K_{16} \end{bmatrix} = \begin{bmatrix} V_1 & 0 & 0 & 0 \\ V_2 & V_3 & 0 & 0 \\ V_4 & V_5 & V_6 & 0 \\ V_7 & V_8 & V_9 & V_{10} \end{bmatrix} \begin{bmatrix} V_1 & V_2 & V_3 & V_4 \\ 0 & V_5 & V_6 & V_7 \\ 0 & 0 & V_8 & V_9 \\ 0 & 0 & 0 & V_{10} \end{bmatrix}$$

[K] $[V^T]$ [V]

CHOLESKY’S ALGORITHM (contd….)

$Ku = P$

$V^T\ Vu = P$

$V^T\ w = P$ where $(w = Vu)$

$$\begin{bmatrix} V_1 & 0 & 0 & 0 \\ V_2 & V_3 & 0 & 0 \\ V_4 & V_5 & 0 & 0 \\ V_7 & V_8 & V_9 & V_{10} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix}$$

$w_1 = P_1/V_1$
$w_2 = (P_2 - V_2 w_1)/V_3$

Similarly we can find w_3, $w_4, \ldots\ldots w_N$

Further, $Vu = w$

$$\begin{bmatrix} V_1 & V_2 & V_3 & V_4 \\ 0 & V_5 & V_6 & V_7 \\ 0 & 0 & V_8 & V_9 \\ 0 & 0 & 0 & V_{10} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}$$

On similar lines, we can find $u_N, u_{N-1}, \ldots\ldots u_1$

HOW TO OBTAIN [V] THE UPPER TRIANGULAR MATRIX

$$V_{11} = \sqrt{K_{11}}$$

$$V_{1i} = k_{1i}/V_{11}$$

$$V_{ii} = \sqrt{[(K_{ii} - \sum_{m=1}^{i-1} V^2_{mi})]} \qquad i>1$$

$$V_{ij} = (k_{ij} - \sum_{m=1}^{i-1} V_{mi}V_{mj})/V_{ii} \qquad j>i$$

$$V_{ij} = 0 \qquad for\ i>j$$

V & *V*^{T} both will be band matrices, with same bandwidth as $[k]_{pp.}$ We can over write '*V*' on $[k]_{pp}$ *. Hence, no need to create a new matrix.*

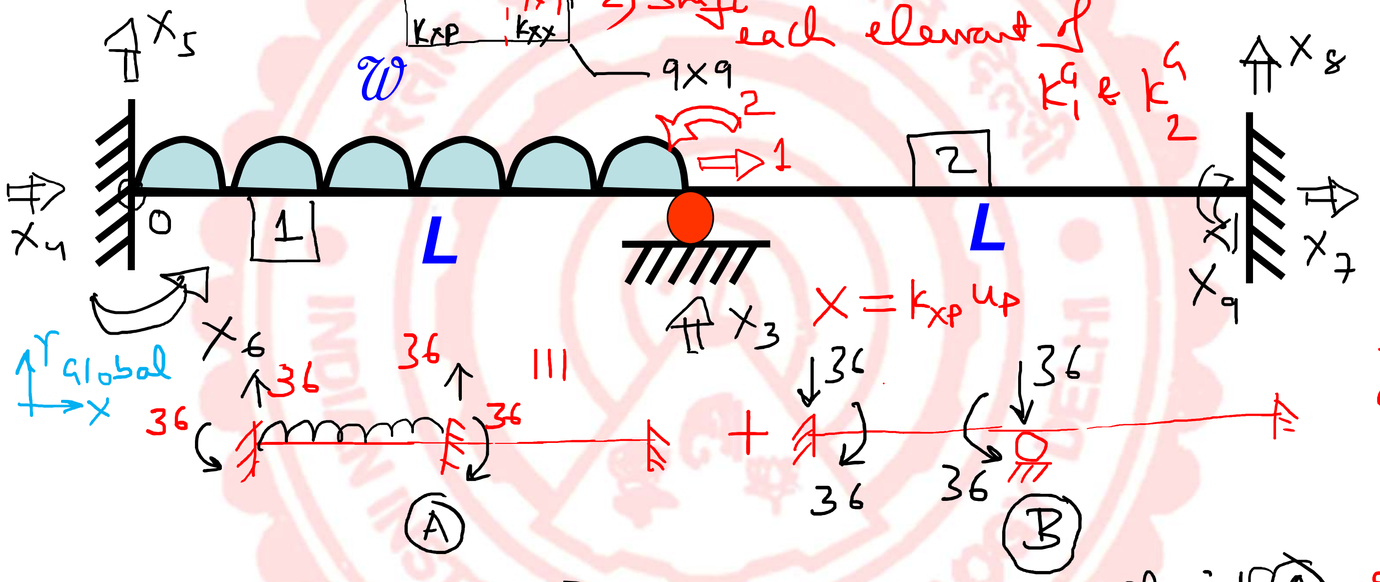
INTERACTIVE EXERCISE

FORM ALL MATRICES FOR THE STRUCTURE

$K_{TS} = \left[\begin{array}{c|c} K_{pp} \; (2\times2) & K_{px} \\ \hline K_{xp} & K_{xx} \; (7\times7) \end{array}\right]$ 9x9

1) Initialize $K_{TS} = 0$
2) Shift each element of K_1^G & K_2^G

w = 12 kN/m, L = 6m $\begin{Bmatrix} P \\ X \end{Bmatrix}$



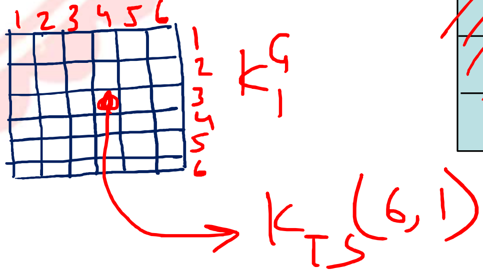
1	0	{P}
2	36	
3	$X_3 - 36$	{X}
4	X_4	
5	$X_5 - 36$	
6	$X_6 - 36$	
7	X_7	
8	X_8	
9	X_9	

We analyse B & superimpose results in A

$$A[\text{Mem } 1] = \begin{array}{cccccc} 4 & 5 & 6 & 1 & 3 & 2 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

$$A[\text{Mem } 2] = \begin{bmatrix} 1 & 3 & 2 & 7 & 8 & 9 \end{bmatrix}$$

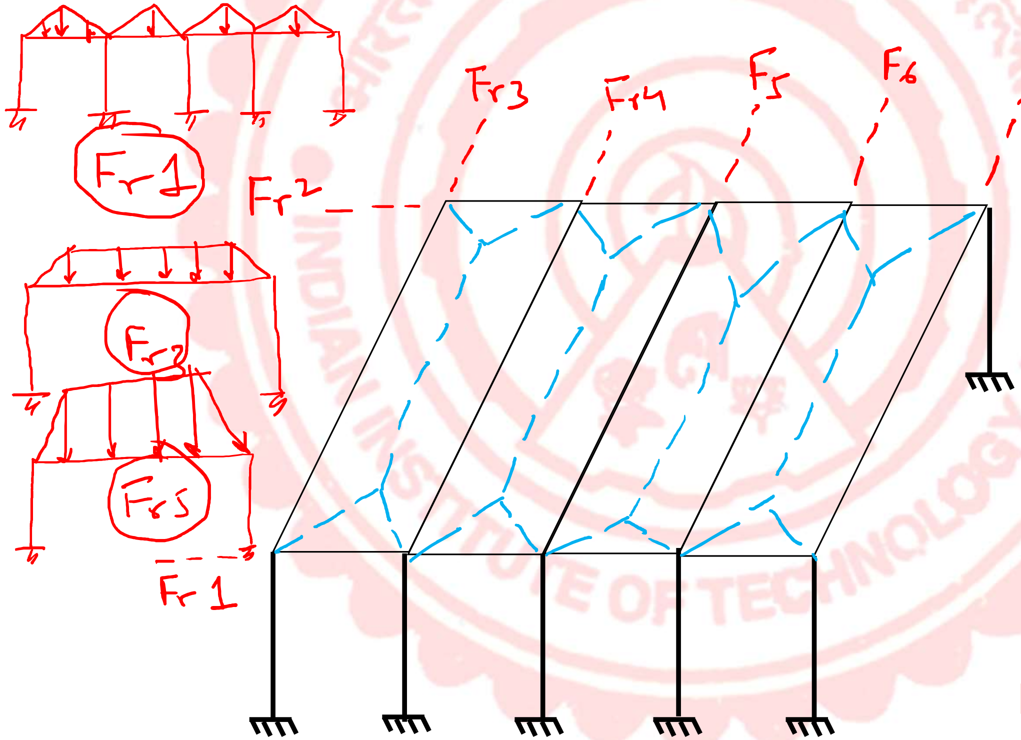
$P = K_{pp} u_p$

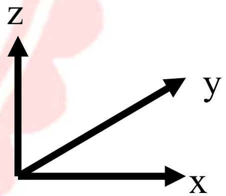


HOW TO ANALYSE 3D STRUCTURES

Option1: A space frame can be broken down into plane frames.

Three frames to be analyzed F_1, F_3, F_5





Member forces to be superimposed for columns

OPTION 2: MATRIX FORMULATIONS FOR 3D STRUCTURES

Alternatively, the 2D formulations can be extended into 3D

1) K_L 12x12

2) I_y & I_z

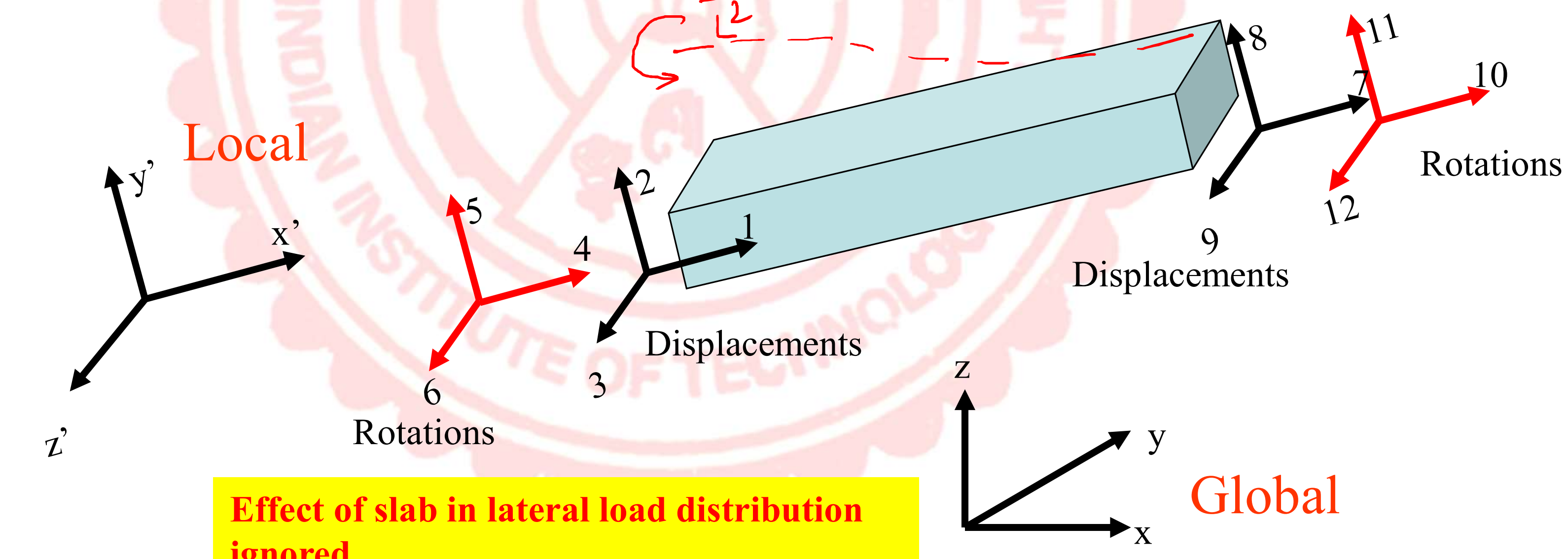
3) Torsion

Additional term (dof 4, 10)

Torsional constant

$$T = \left[\frac{GJ}{L}\right]\theta \Leftarrow d_{10} - d_4$$

$\frac{6EI_z}{L^2}$

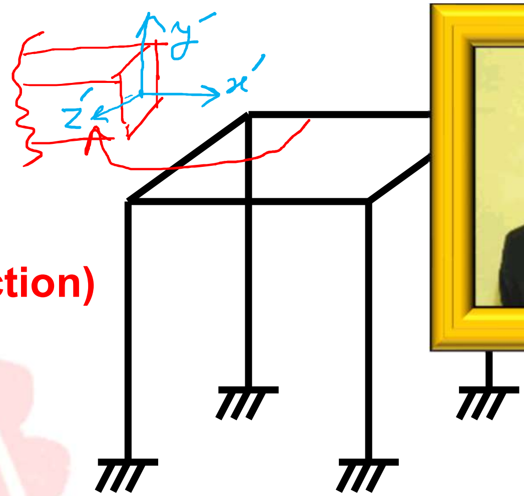


Effect of slab in lateral load distribution ignored…..

$$
[K]_L = \begin{array}{c}
1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12
\end{array}
\begin{bmatrix}
\frac{EA}{L} & & & & & & & & & & & \\
0 & \frac{12EI_z}{L^3} & & & & & & & & & & \\
0 & 0 & \frac{12EI_y}{L^3} & & & & & & & & & \\
0 & 0 & 0 & \frac{GJ}{L} & & & & & & & & \\
0 & 0 & -\frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & & & & & & & \\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L} & & & & & & \\
-\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 & \frac{EA}{L} & & & & & \\
0 & -\frac{12EI_z}{L^3} & 0 & 0 & 0 & -\frac{6EI_z}{L^2} & 0 & \frac{12EI_z}{L^3} & & & & \\
0 & 0 & -\frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 & 0 & 0 & \frac{12EI_y}{L^3} & & & \\
0 & 0 & 0 & -\frac{GL}{L} & 0 & 0 & 0 & 0 & 0 & \frac{GJ}{L} & & \\
0 & 0 & -\frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & \\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L}
\end{bmatrix}
$$

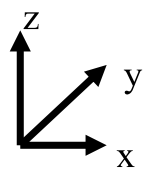
GJ: $= I_p$ for circular sections; $= J$ for non-circular sections

x' → along centroidal axis of the member.
z'→ towards viewer. or vice versa
y'→ can be ascertained by right hand system rule
(y' and z' should be along the principal axes of cross section)



$$\hat{k} = \hat{i} x \hat{j}$$

x' : (1, 0, 0)
y' : (0, 1, 0)
z' : (0, 0, 1)



COORDINATE TRANSFORMATION

For 2D
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l_1 & m_1 \\ l_2 & m_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

l_1, m_1 : direction cosines of x' axis w.r.t global system.
l_2, m_2 : direction cosines of y' axis w.r.t. global system.

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Direction cosines

l_1, m_1, n_1 : x'
l_2, m_2, n_2 : y'
l_3, m_3, n_3 : z'

R

Member Transformation matrix

[T] =

12 x 12

Disp1	1 2 3	R	0	0	0
Rot 1	4 5 6	0	R	0	0
Dis2	7 8 9	0	0	R	0
Rot2	10 11 12	0	0	0	R

$L = [(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2]^{1/2}$
$l_1 = (x_2-x_1)/L$
$m_1 = (y_2-y_1)/L$
$n_1 = (z_2-z_1)/L$

Unit vector along x' $= l_1\hat{i} + m_1\hat{j} + n_1 k = \hat{i}'$

We must specify the direction of y-axis

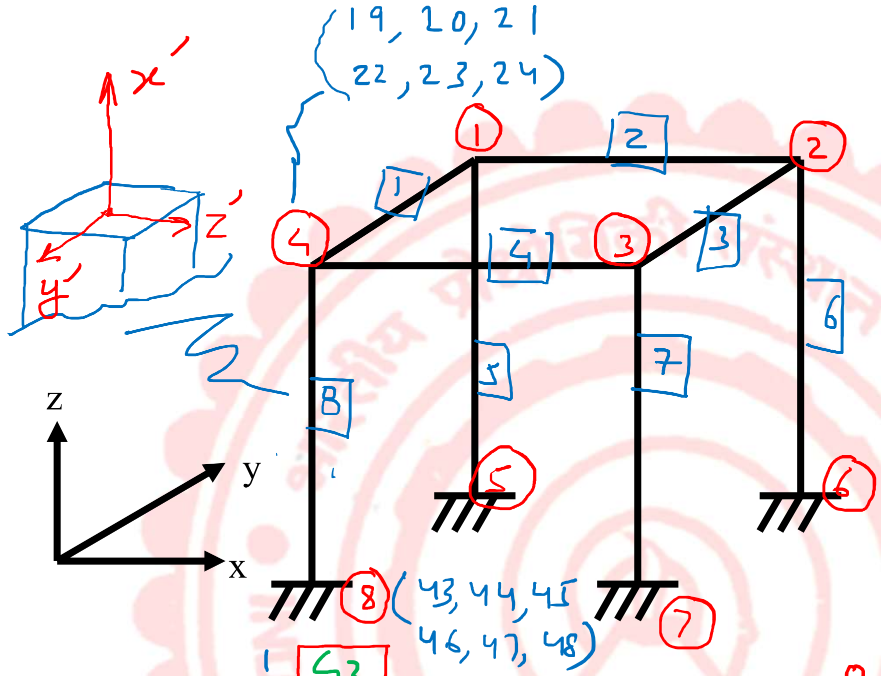
1. Unit vector along y' $= l_2\hat{i} + m_2\hat{j} + n_2 k = \hat{j}'$

OR

2. Two points along y' : (x_3,y_3,z_3) & (x_4,y_4,z_4) so that we can find: l_2, m_2, n_2

Since *x'y'z'* from right handed coordinate system, unit vector along z'

$$\hat{k}' = \hat{i}' x \hat{j}' = l_3\hat{i} + m_3\hat{j} + n_3\hat{k}$$



$$K_{TS} = \left[\begin{array}{c|c} 24\times24 & \\ \hline & 24\times24 \end{array}\right] \quad 48\times48$$

$$R^{8} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$T^{8} = \begin{bmatrix} R & & & 0 \\ & R & & \\ & & R & \\ 0 & & & R \end{bmatrix}$$

$$K_G^{8} = T^{T} K_L^{8}\, T = \langle 12\times12 \rangle$$

A(Mem 8)

1	43
2	44
3	45
4	46
5	47
6	48
7	19
8	20
9	21
10	22
11	23
12	24

$$K^{G8}(6,6) \longrightarrow K_{TS}(48,48)$$

K_{PP} — half banded

K_{xx} — full

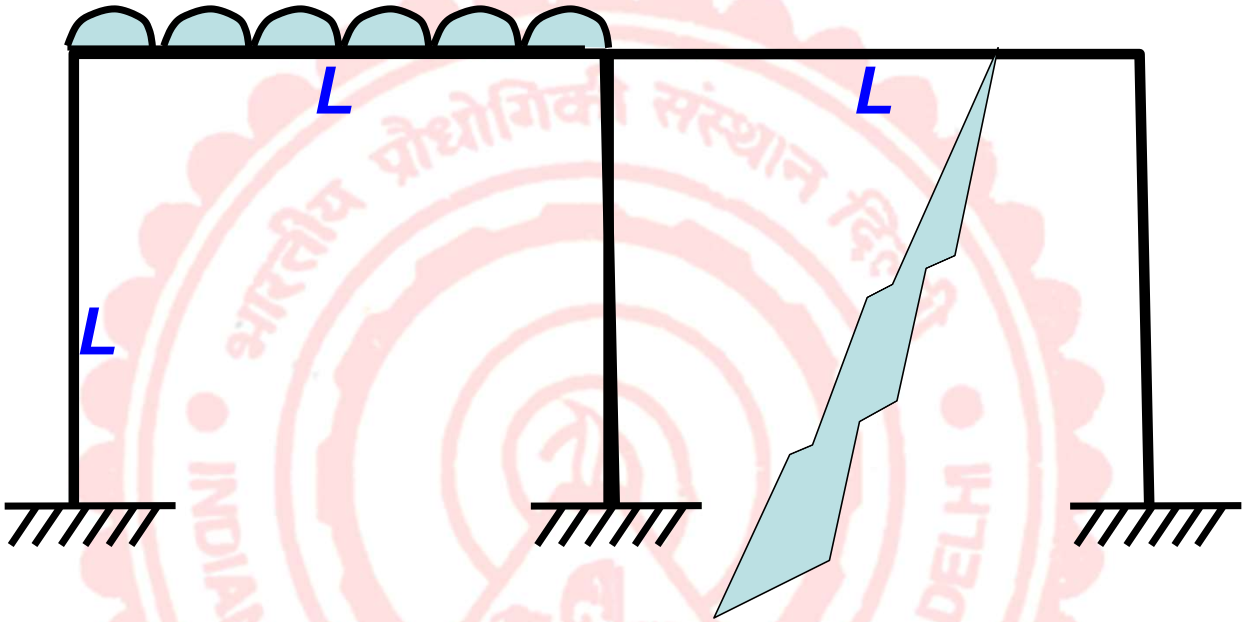
K_{px} or K_{xp} → full

ANALYSIS STEPS

- Fix member dimensions tentatively.
- Perform analysis.
- Check for adequacy of member sizes at key locations.
- Revise dimensions if necessary

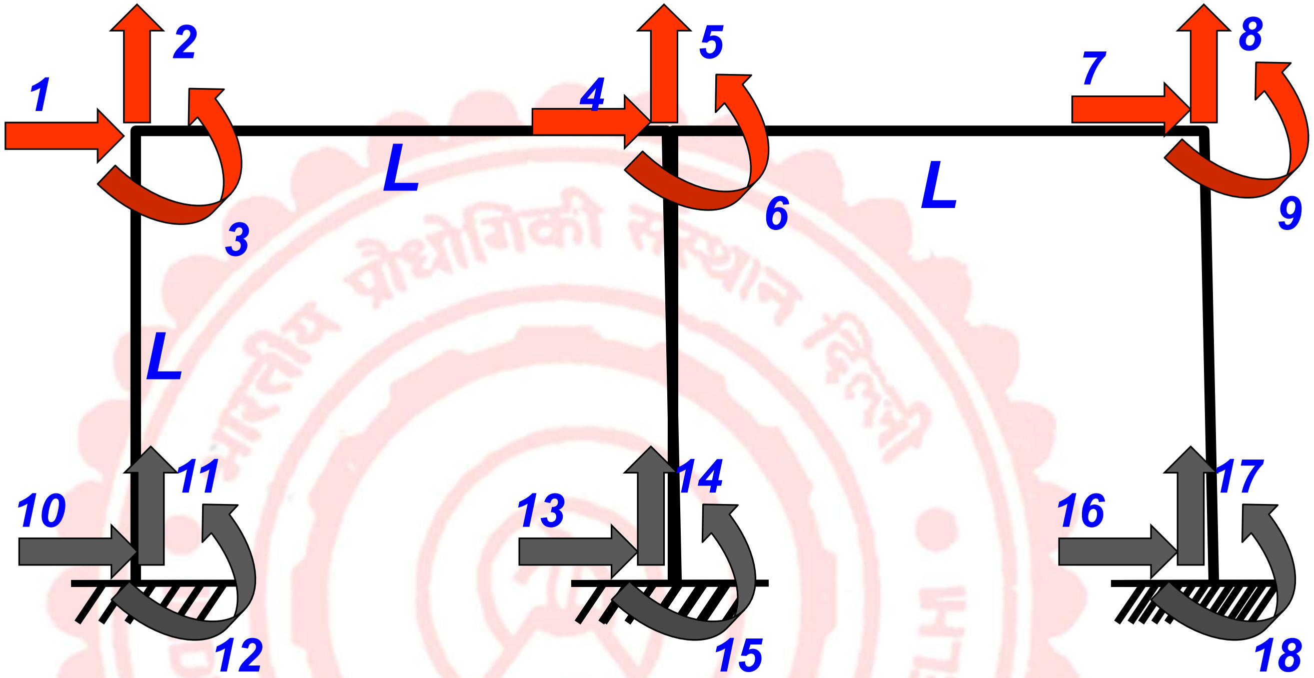
& reanalyze

TEMPERATURE VARIATION

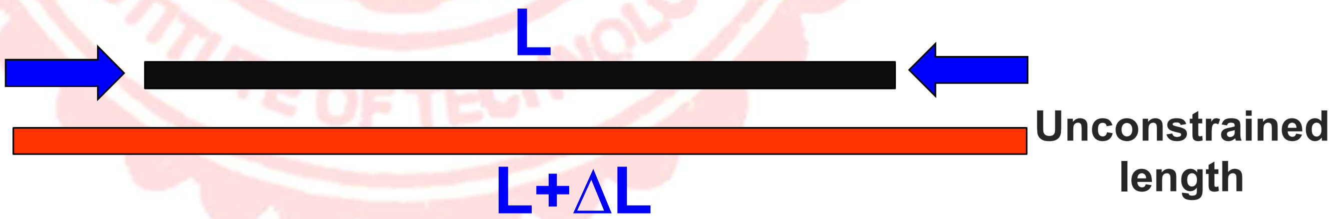


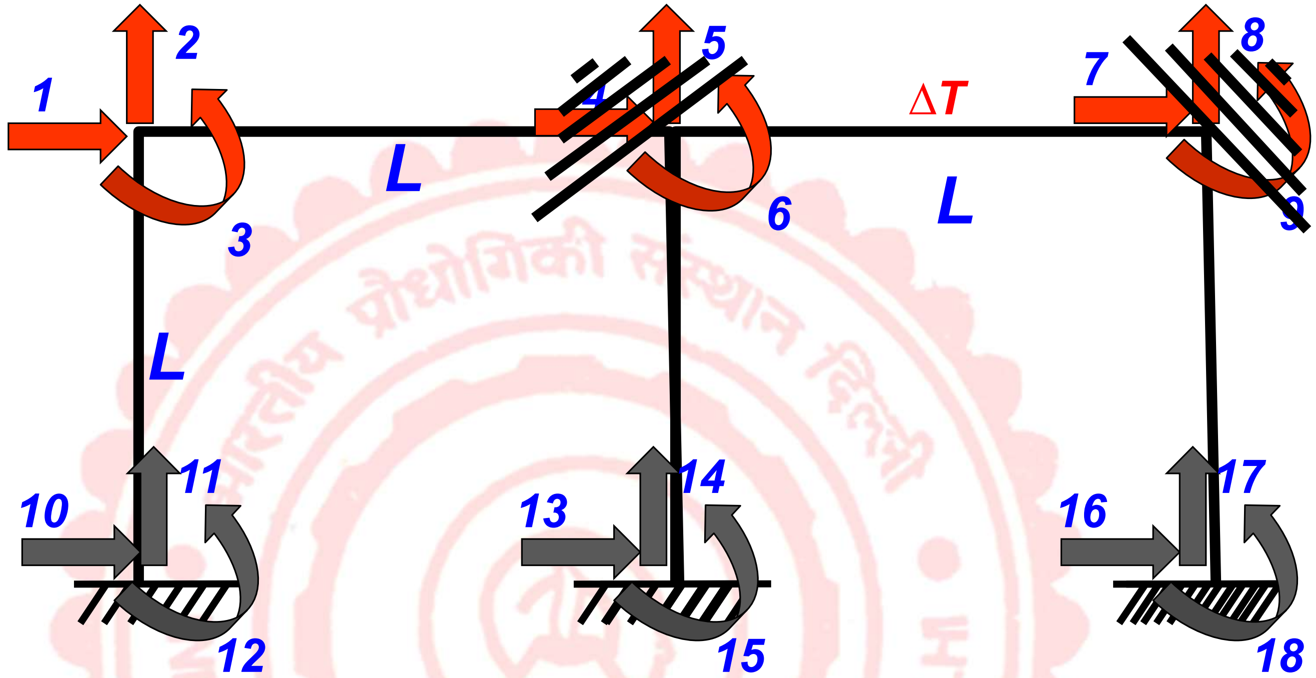
Let there be uniform temperature change ΔT throughout the member

Determinate vs indeterminate structures, any difference??



Identify and mark the degrees of freedom

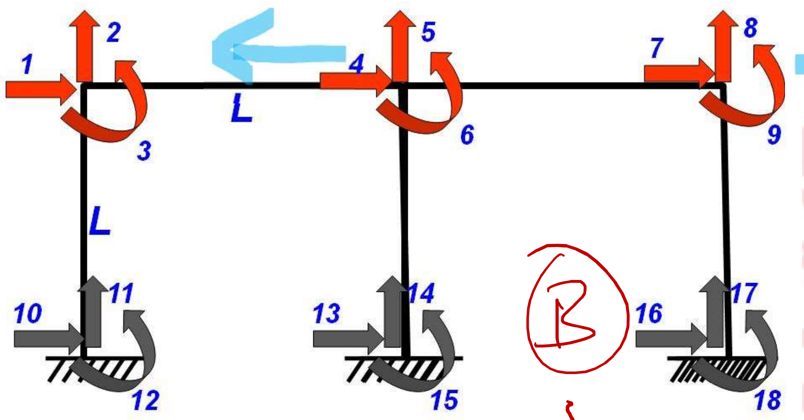




Convert thermal effect into fixed ended forces

$$F^f = \left(\frac{EA}{L}\right)\Delta L \qquad \Delta L = L\alpha\Delta T \qquad F^f = EA\alpha\Delta T$$

APPLY OPPOSITE OF THE FEF ON THE STRUCTURE



$$F^f = EA\alpha\Delta T$$

A

$$P = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -EA\alpha\Delta T \\ 0 \\ 0 \\ EA\alpha\Delta T \\ 0 \\ 0 \end{bmatrix}$$

Solve the matrix equation as before and obtain displacements and member forces

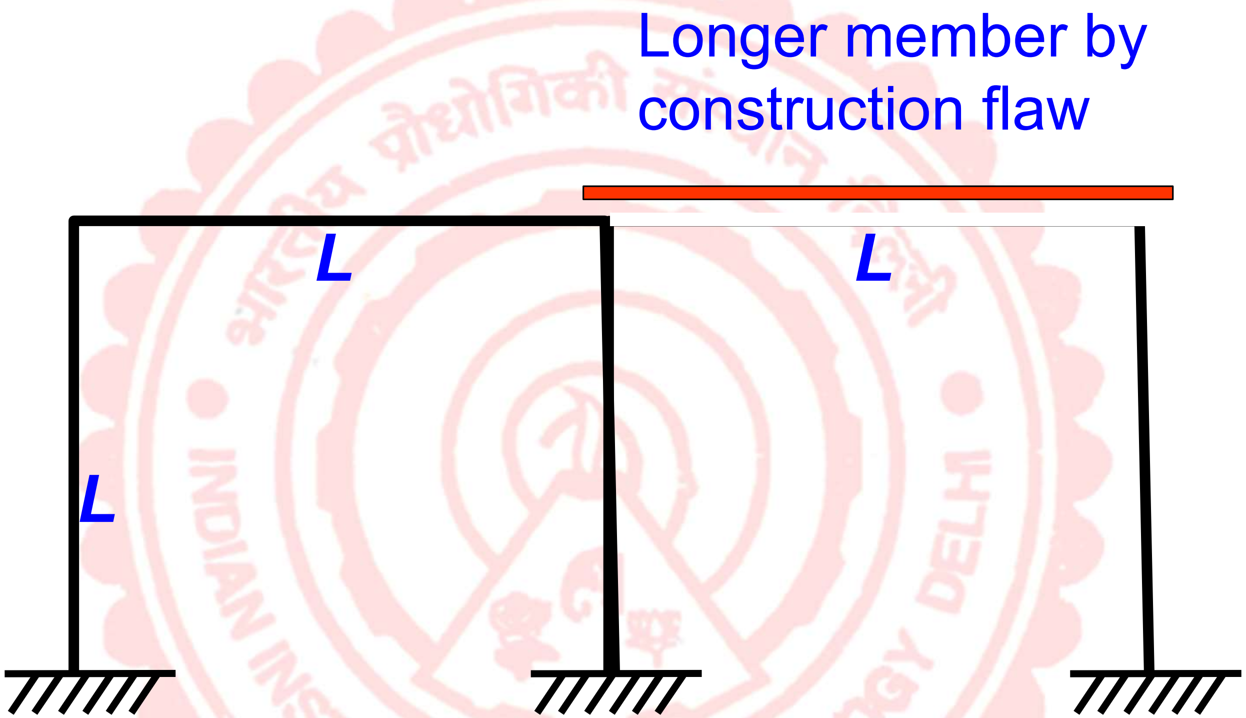
Final member end forces can be obtained by superimposing the fixed ended condition with above solution

$$\{f\} = \{K_L\}\{d\} + \{f\}^F$$

B A

What would happen in case of temperature fall??

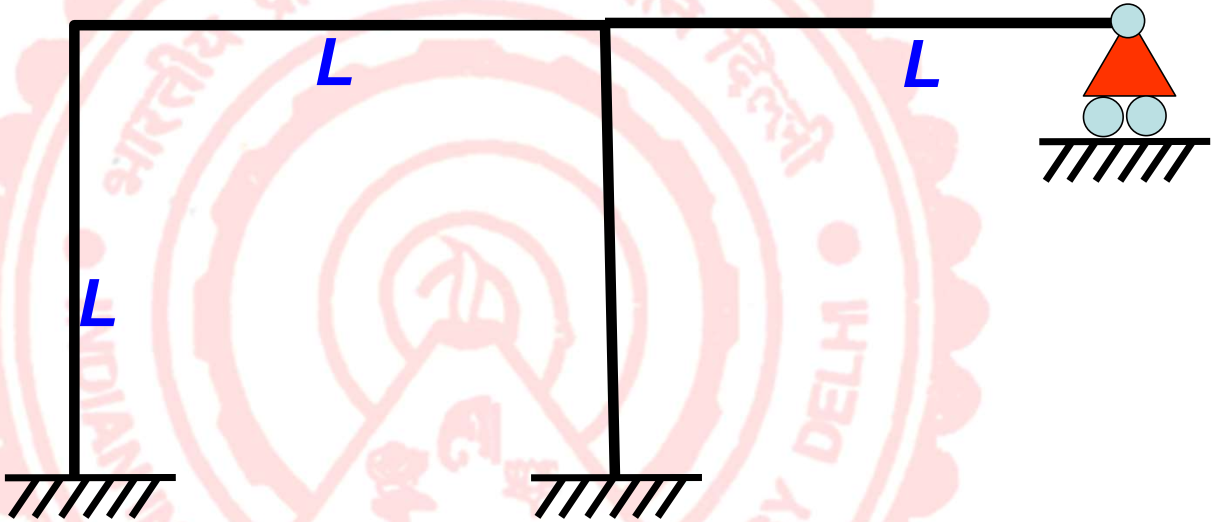
EXTENSION TO LACK OF FIT



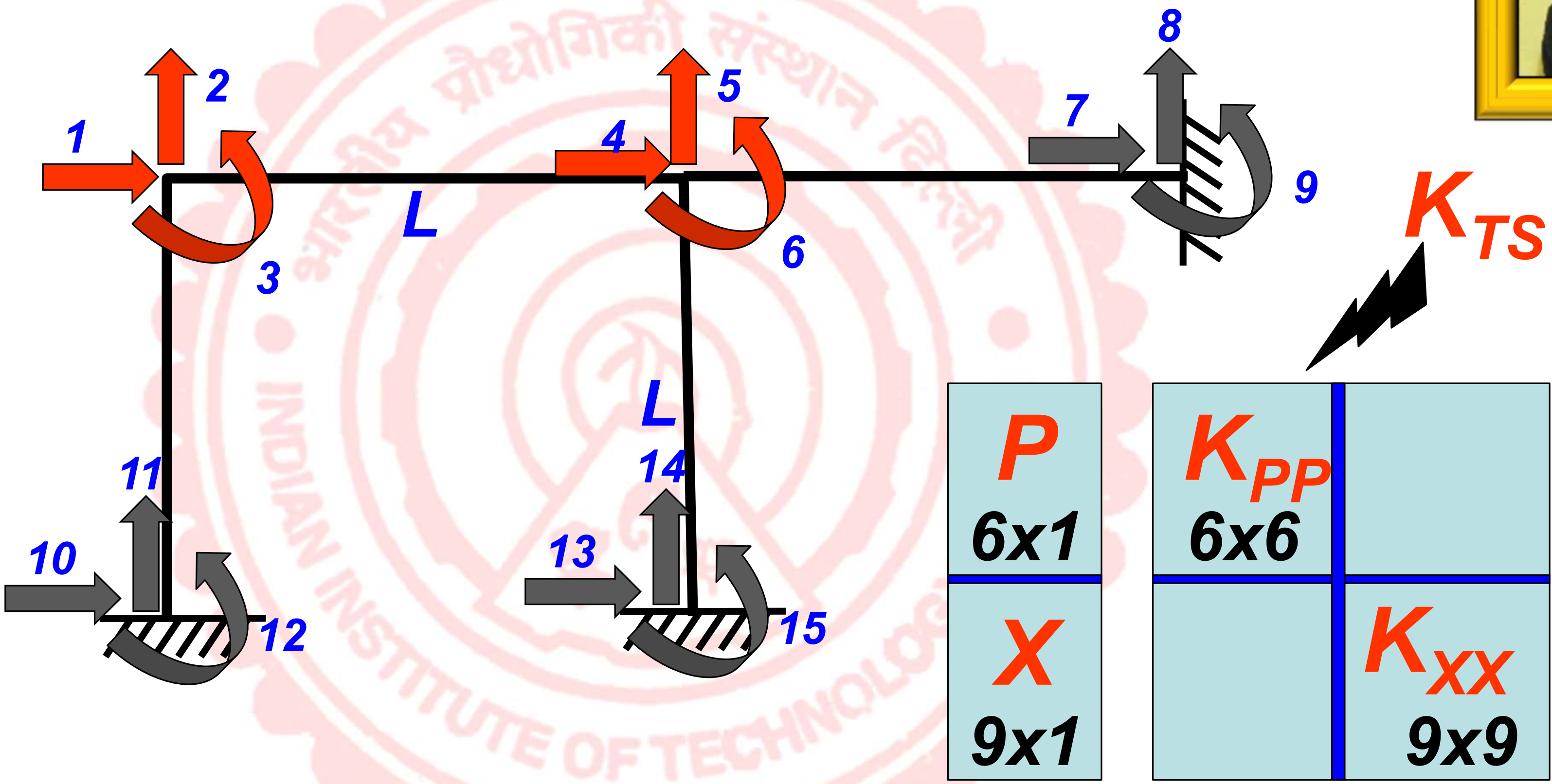
Longer member : Analogous to temperature rise

Smaller member : Analogous to temperature fall

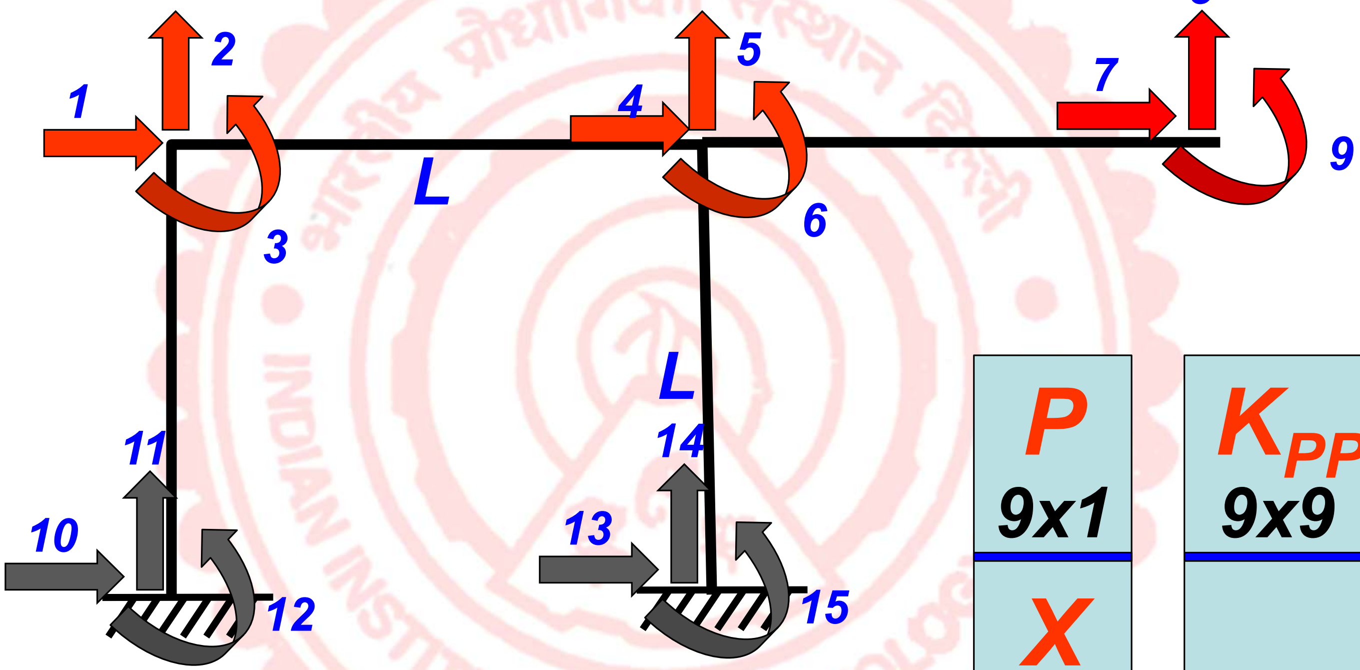
HINGED/ GUIDED SUPPORTS

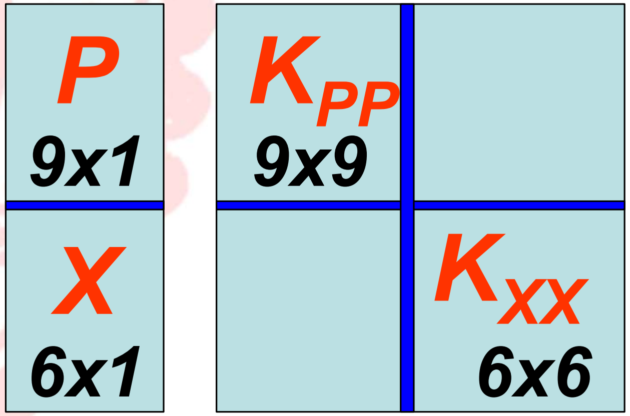


Let us first consider all supports to be fully rigid as treated so far……



Let us now altogether remove the right support……

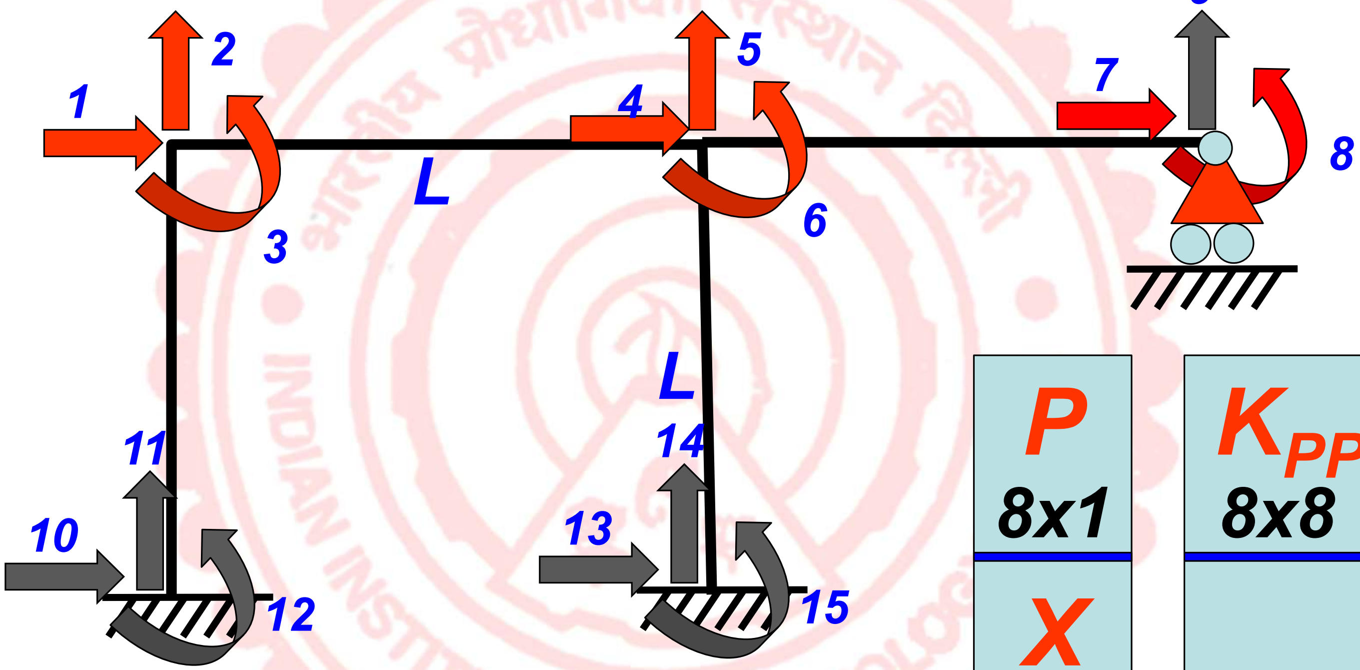


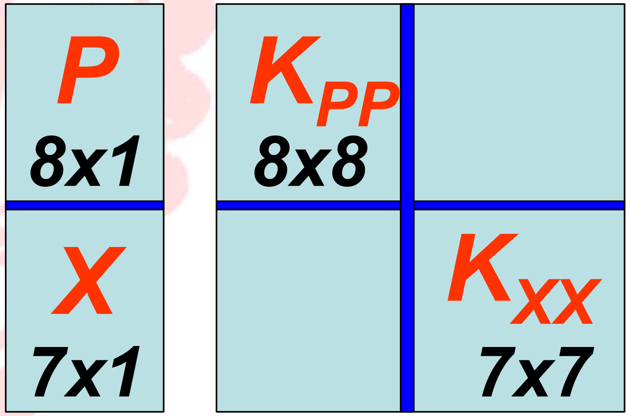


$$P_7 = P_8 = P_9 = 0$$

Let us now introduce the hinge

Renumbering

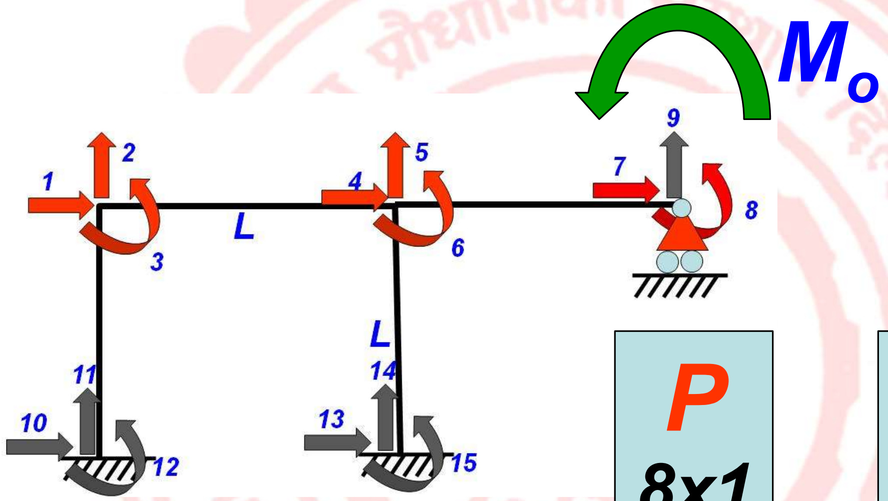




$P_7 = P_8 = 0$ (if no load acting at those points)

$u_7 \neq 0$ and $u_8 \neq 0$

WHAT HAPPENS WHEN THERE IS A LOAD ACTING ALONG THE RELEASE?



$f^6 = M_o$

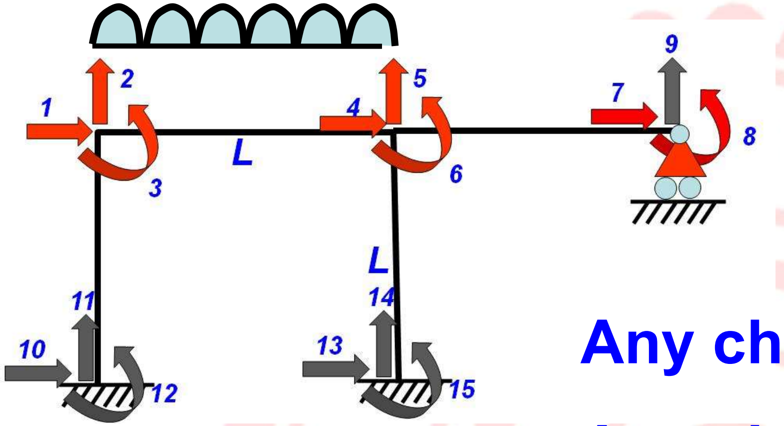
$$\begin{bmatrix} \underset{8\times1}{P} \\ \underset{7\times1}{X} \end{bmatrix} = \begin{bmatrix} \underset{8\times8}{K_{PP}} & K_{PX} \\ K_{XP} & \underset{7\times7}{K_{XX}} \end{bmatrix} \begin{bmatrix} \underset{8\times1}{u_P} \\ \underset{7\times1}{u_x} \end{bmatrix}$$

How would P change ?

$P_8 = M_o$

After solution, corresponding member end moment = M_o

UDL IN ADJOINING SPAN



Convert the UDL into equivalent joint loads

Any change in K_{TS} ? YES/ NO

Any change in P, X? YES/ NO

$$\begin{bmatrix} P \\ 8x1 \\ \hline X \\ 7x1 \end{bmatrix} = \left[\begin{array}{c|c} K_{PP} & K_{PX} \\ 8x8 & \\ \hline K_{XP} & K_{XX} \\ & 7x7 \end{array}\right] \begin{bmatrix} u_P \\ 8x1 \\ \hline u_x \\ 7x1 \end{bmatrix}$$

Upon solving, we will get the values of u_7 and u_8

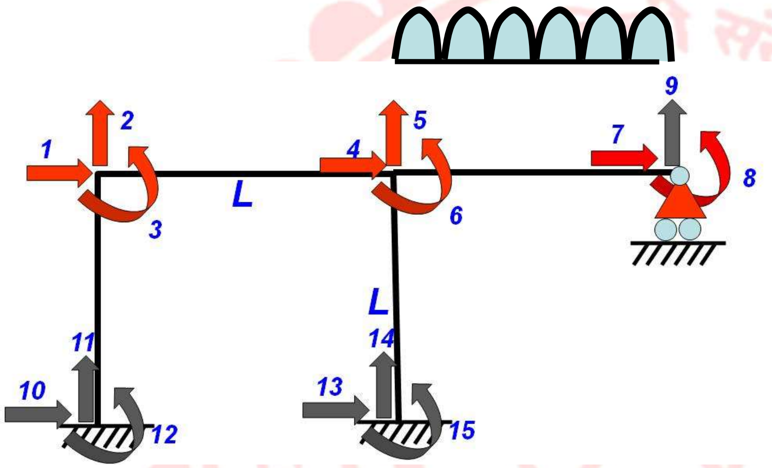
$$P_2 = P_5 = -\frac{wL}{2}$$

$$P_3 = ?? \quad -\frac{wL^2}{12}$$

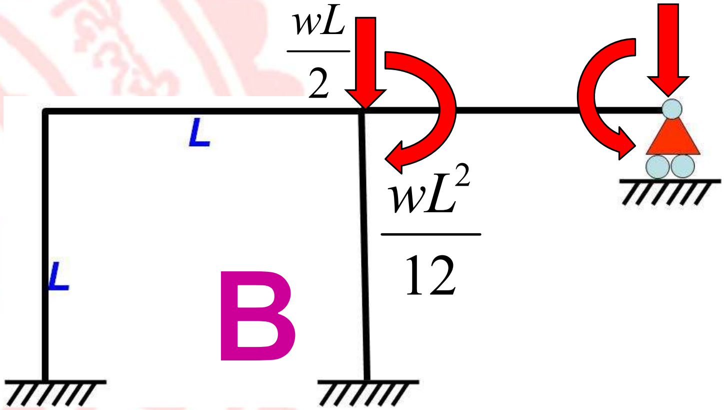
$$P_6 = ?? \quad +\frac{wL^2}{12}$$

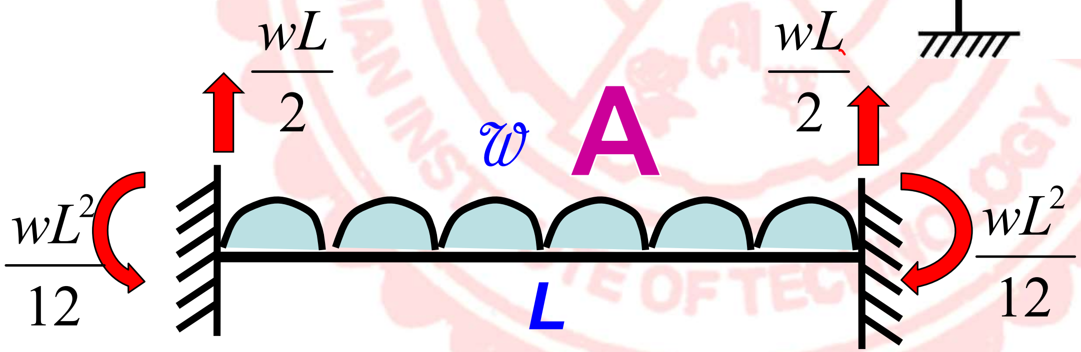
EXERCISE: Form the matrices P and X

WHEN EQUIVALENT JOINT LOAD HAS COMPONENT ALONG THE RELEASE



Convert UDL into equivalent joint loads





$P_5 = -wL/2$

$P_6 = -wL^2/12$

$P_8 = +wL^2/12$

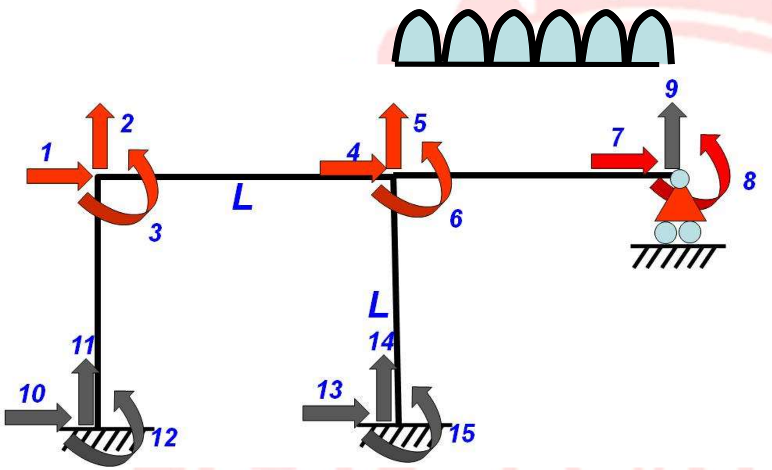
In end, $\{f\} = \{K_L\}\{d\} + \{f\}^F$

Final moment at right end of member = 0

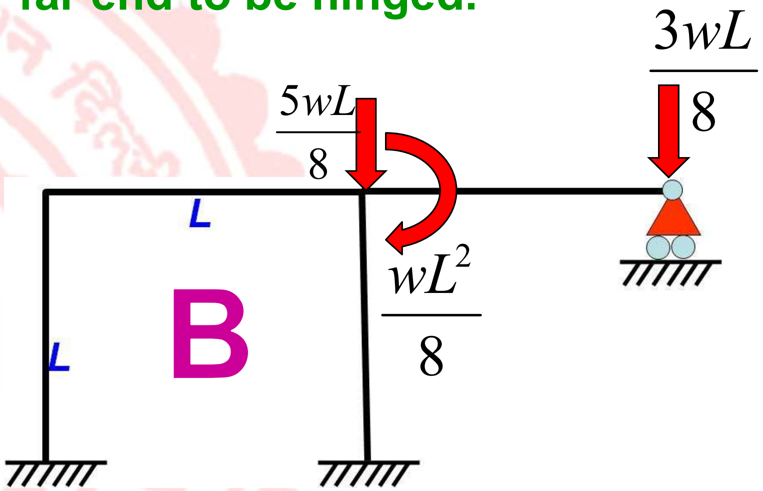
$\frac{wL^2}{12} + \left(-\frac{wL^2}{12}\right)$

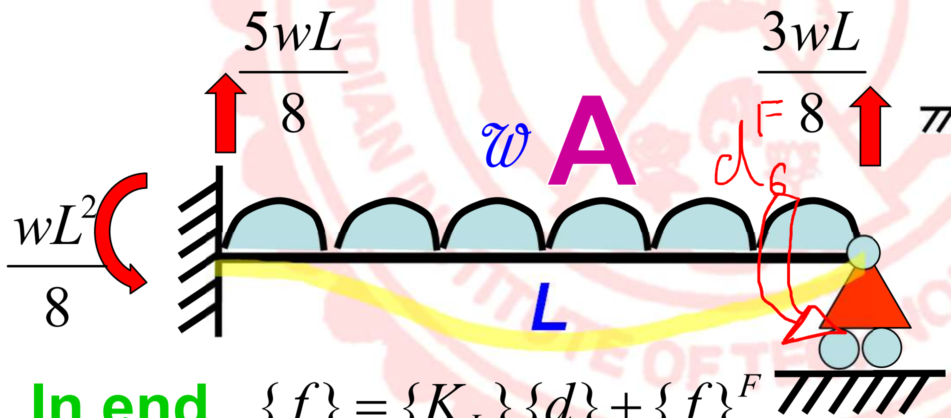
X_9 also gets additional term

ALTERNATE APPROACH



Convert UDL into equivalent joint loads, but considering the far end to be hinged.





$P_5 = -5wL/8$

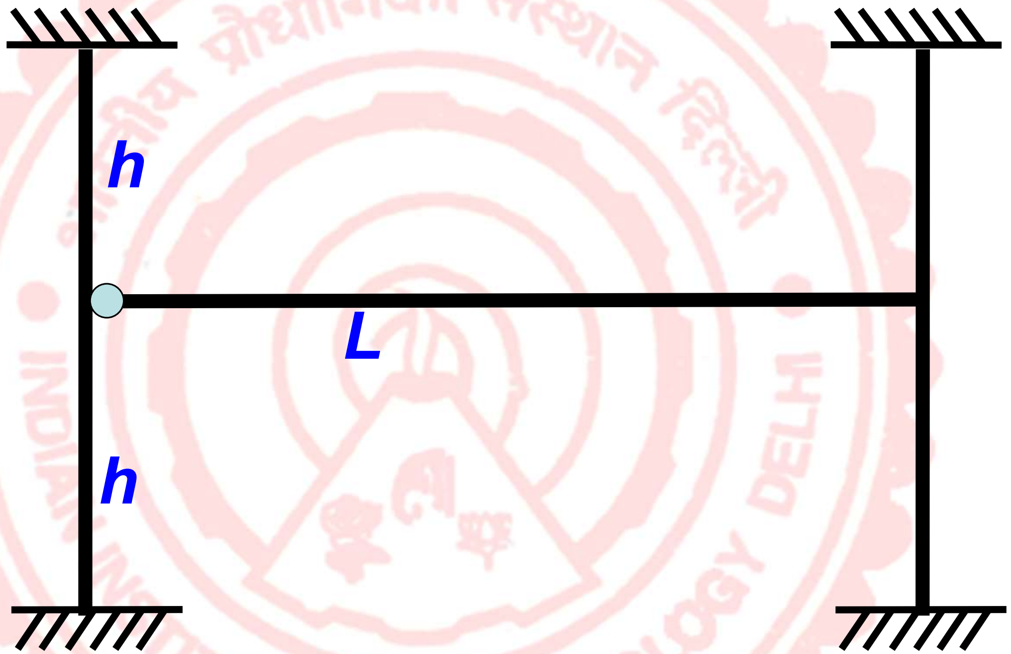
$P_6 = -wL^2/8$

$P_8 = 0$

In end, $\{f\} = \{K_L\}\{d\} + \{f\}^F$

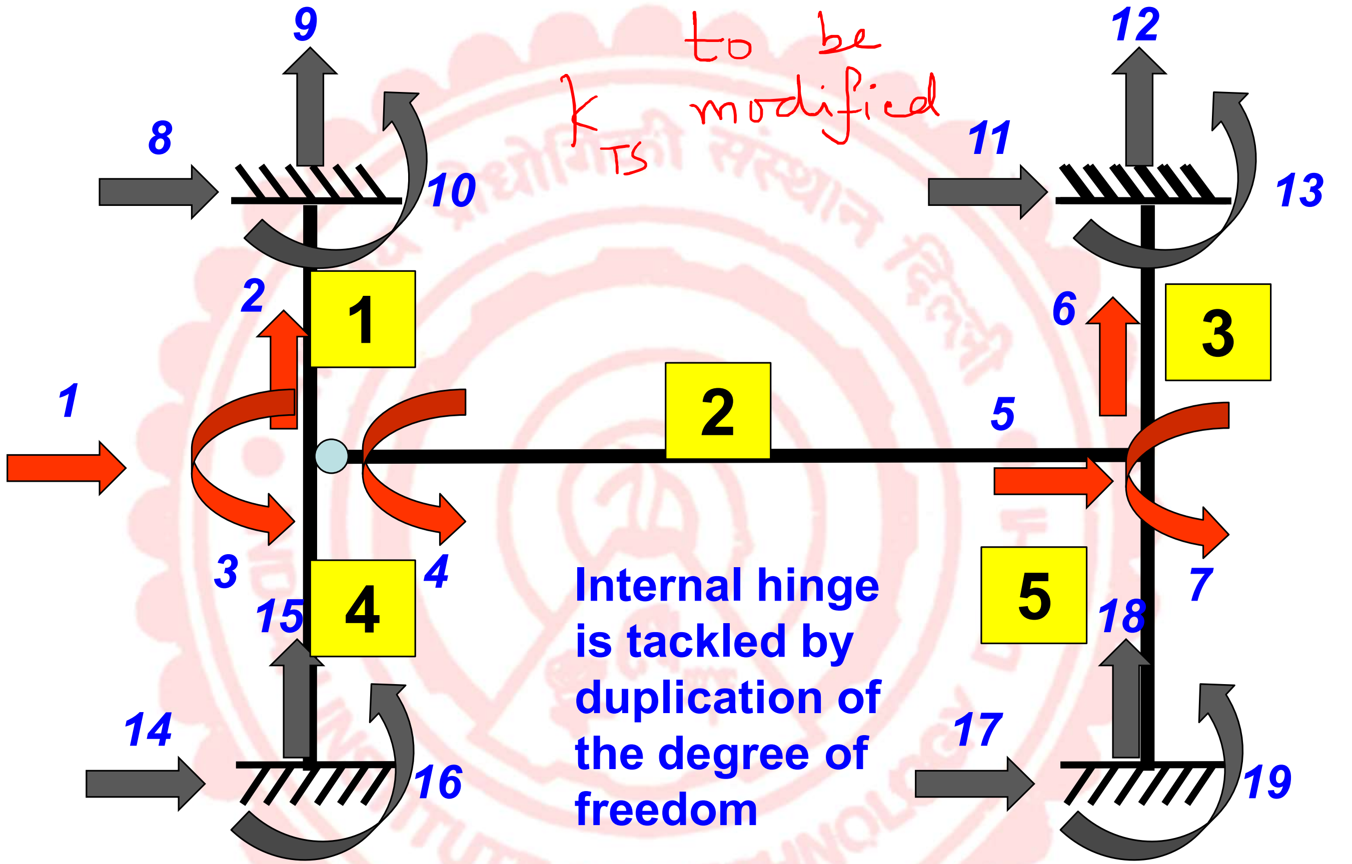
Caution: **Displacement correction needed in end. d^f to be added to the displacement from the output**

INTERNAL HINGE



Hinge in beam only (not in column)

INTERNAL HINGE



to be modified K_{TS}

Internal hinge is tackled by duplication of the degree of freedom

DOF (7) : Common for Members 2, 3, 5

DOF (3) : Common for Members 1, 4

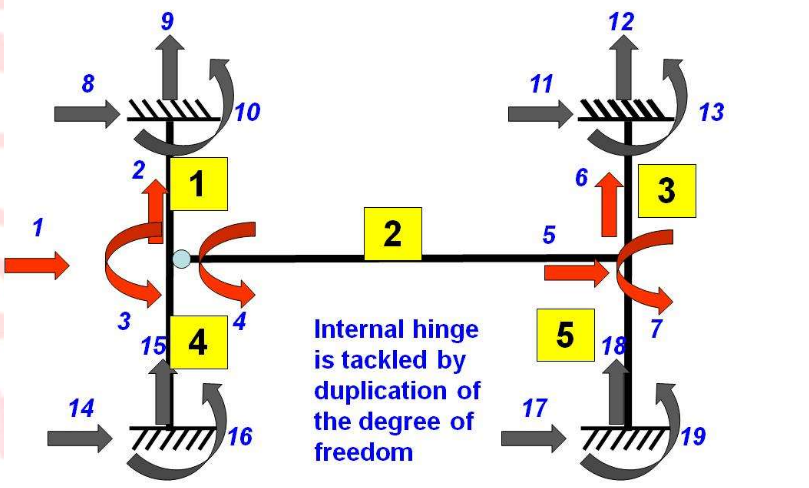
DOF (4) : Member 2 ONLY

INTERNAL HINGE

Terms corresponding to DOF (4) will get contribution from member 2 only.

Take care of the DOF in code number approach format

A[1] = (1, 2, 3, 8, 9, 10)

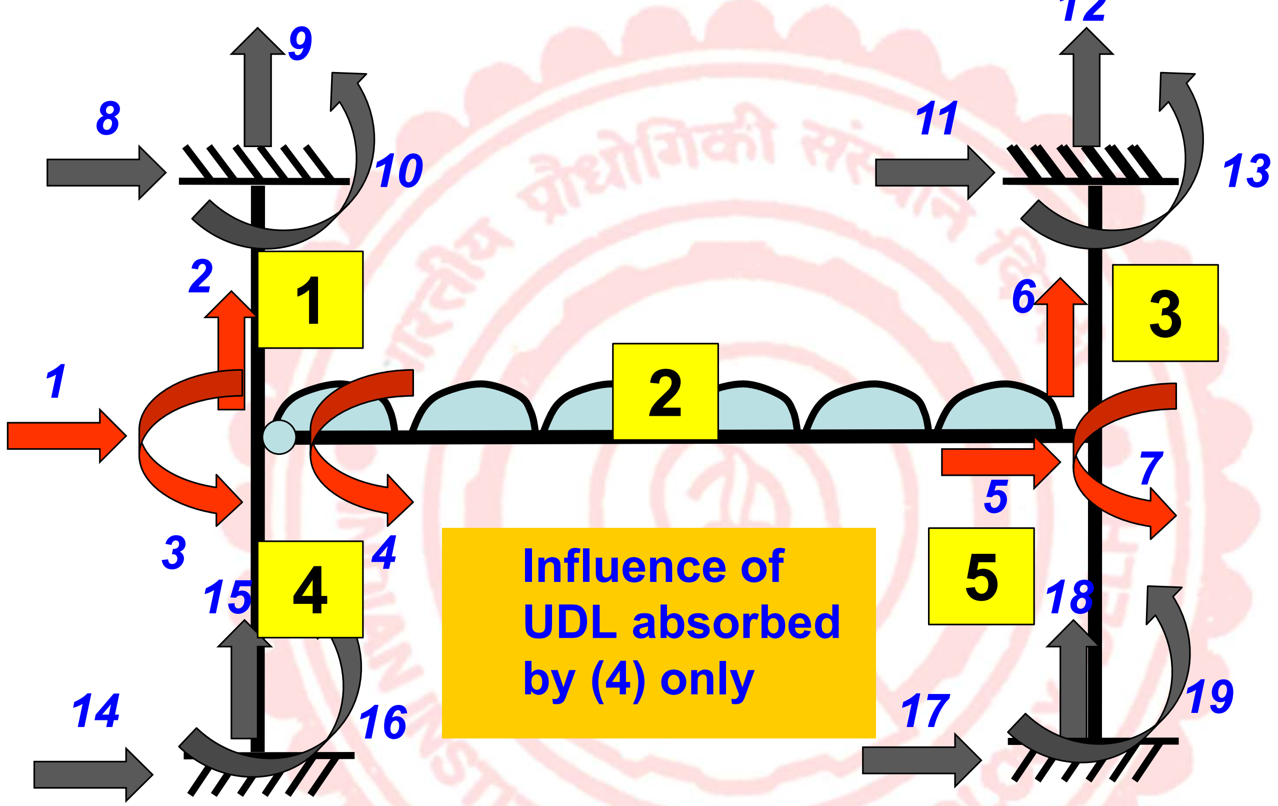


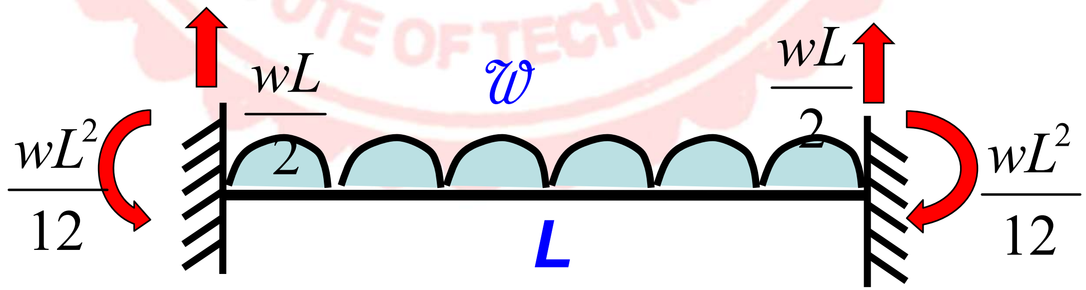
Association matrices:

$$A[2] = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 5 \\ 6 \\ 7 \end{bmatrix} \qquad A[4] = \begin{bmatrix} 14 \\ 15 \\ 16 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} P \; 7x1 \\ X \; 12x1 \end{bmatrix} = \begin{bmatrix} K_{PP} \; 7x7 & K_{PX} \\ K_{XP} & K_{XX} \; 12x12 \end{bmatrix} \begin{bmatrix} u_P \; 7x1 \\ u_X \; 12x1 \end{bmatrix}$$

INTERNAL HINGE WITH UDL





$P_1 = 0$

$P_2 = -wL/2$

$P_3 = 0$

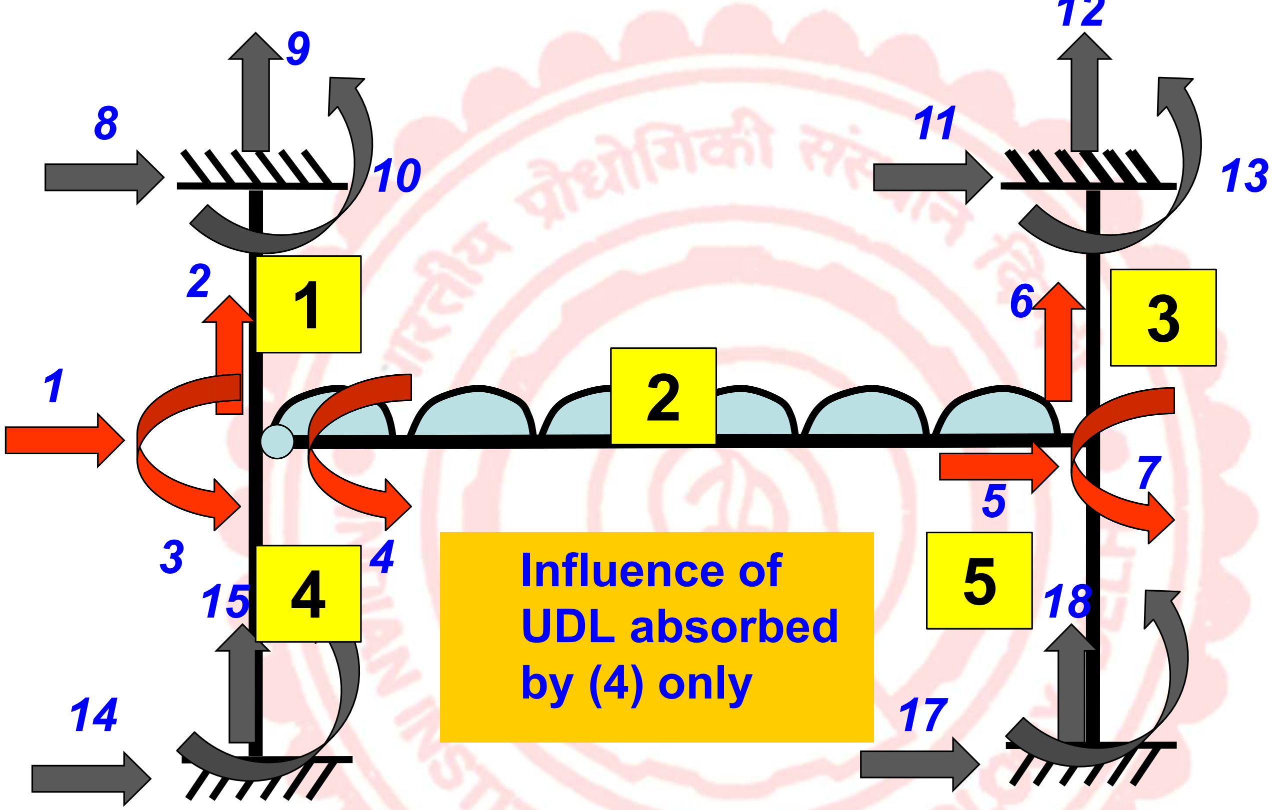
$P_4 = -wL^2/12$

$P_5 = 0$

$P_6 = -wL/2$

$P_7 = +wL^2/12$

INTERNAL HINGE WITH UDL



$P_1 = 0$

$P_2 = -wL/2$

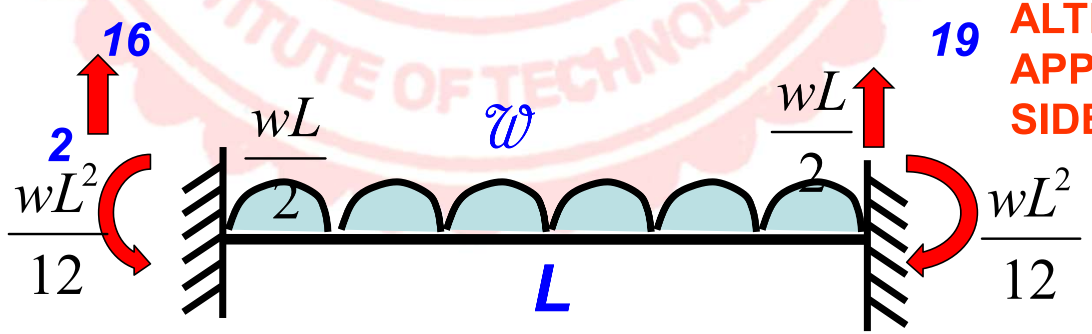
$P_3 = 0$

$P_4 = -wL^2/12$

$P_5 = 0$

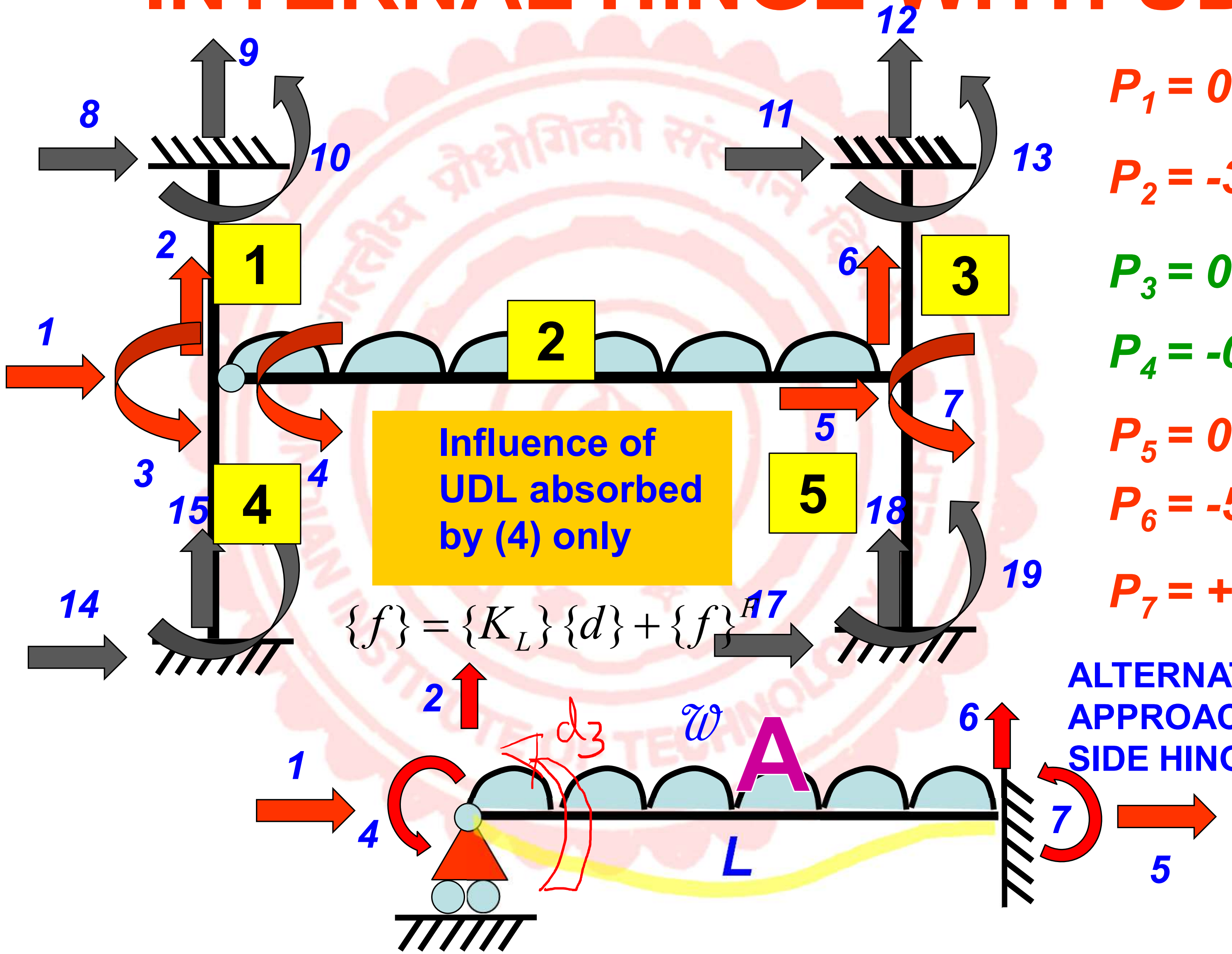
$P_6 = -wL/2$

$P_7 = +wL^2/12$



ALTERNATE APPROACH ONE SIDE HINGED……

INTERNAL HINGE WITH UDL



$$\{f\} = \{K_L\}\{d\} + \{f\}^{R}$$

$P_1 = 0$

$P_2 = -3wL/8$

$P_3 = 0$

$P_4 = -0$

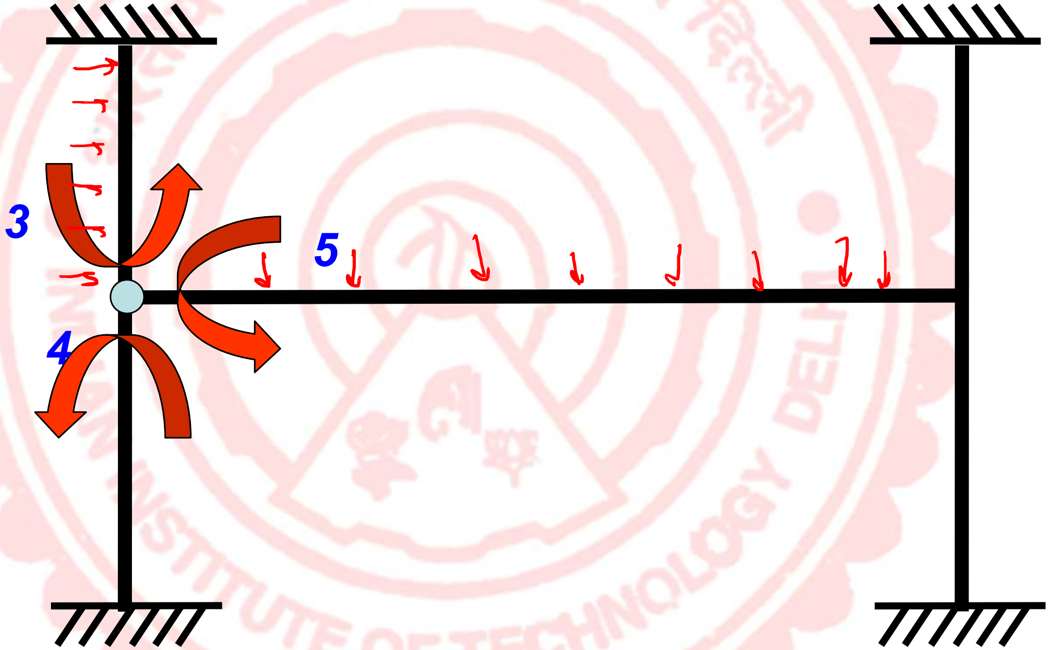
$P_5 = 0$

$P_6 = -5wL/8$

$P_7 = +wL^2/8$

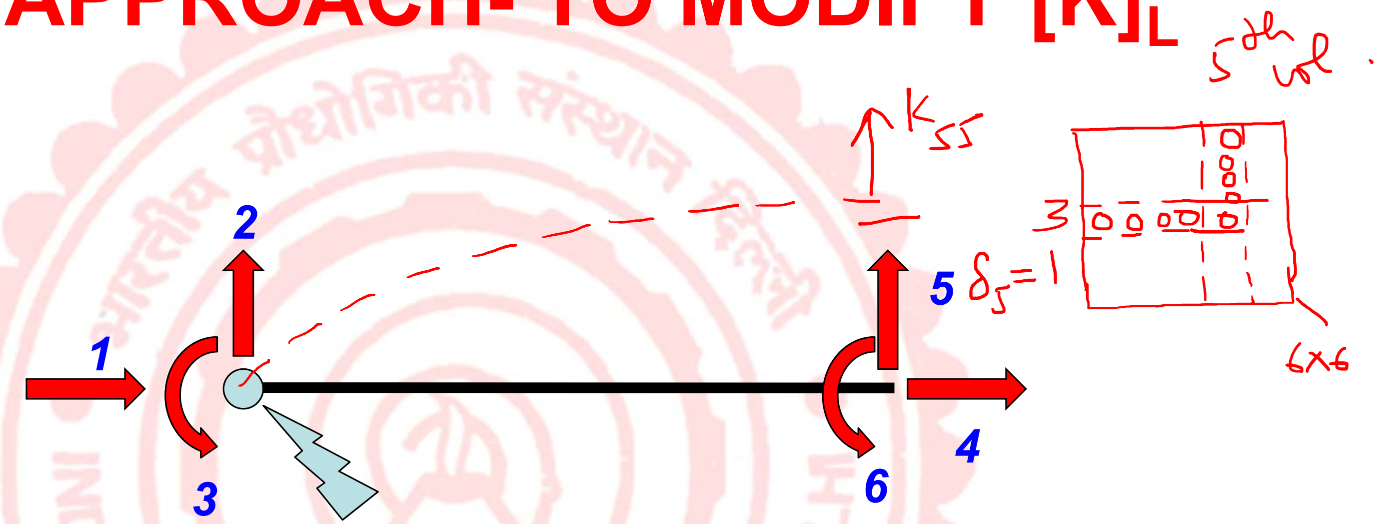
ALTERNATE APPROACH ONE SIDE HINGED……

HINGE THROUGH COLUMN AND BEAM



Independent DOF for all members meeting at the joint, rest of the procedure same.

INTERNAL HINGE : ALTERNATE APPROACH- TO MODIFY $[K]_L$



Permanent hinge

This boundary condition is not to be altered while deriving the member stiffness matrix

Hinge not to be fixed while deriving the member stiffness matrix

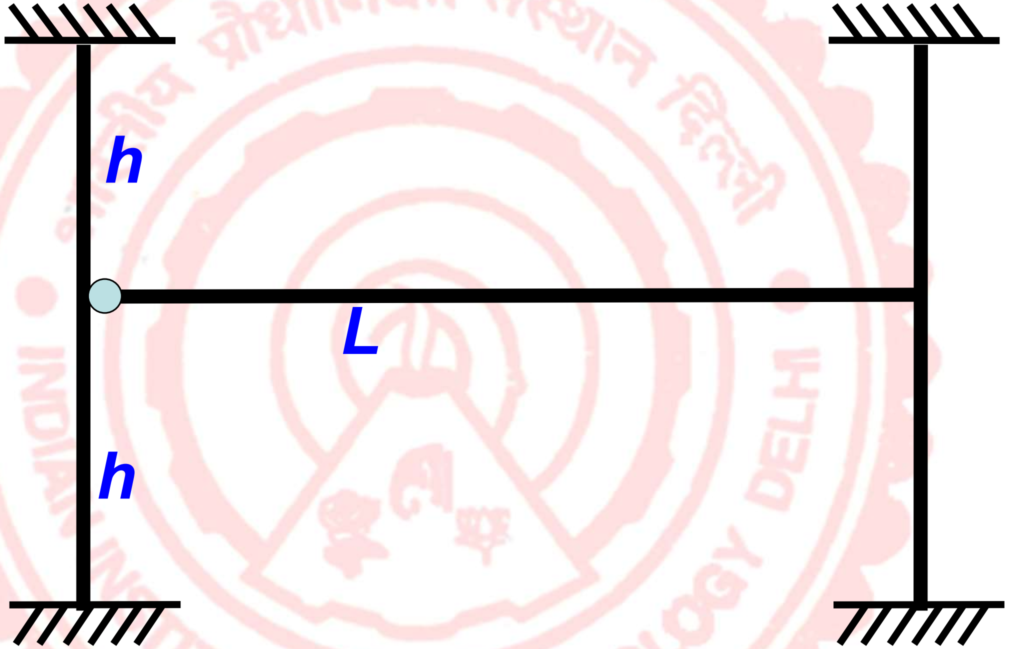
INTERNAL HINGE : ALTERNATE APPROACH- TO MODIFY $[K]_L$

$$[K]_L = \begin{bmatrix} \frac{EA}{L} & & & & & \\ 0 & \frac{3EI}{L^3} & & & & \\ 0 & 0 & 0 & & & \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & & \\ 0 & -\frac{3EI}{L^3} & 0 & 0 & \frac{3EI}{L^3} & \\ 0 & \frac{3EI}{L^2} & 0 & 0 & -\frac{3EI}{L^2} & \frac{3EI}{L} \end{bmatrix}$$

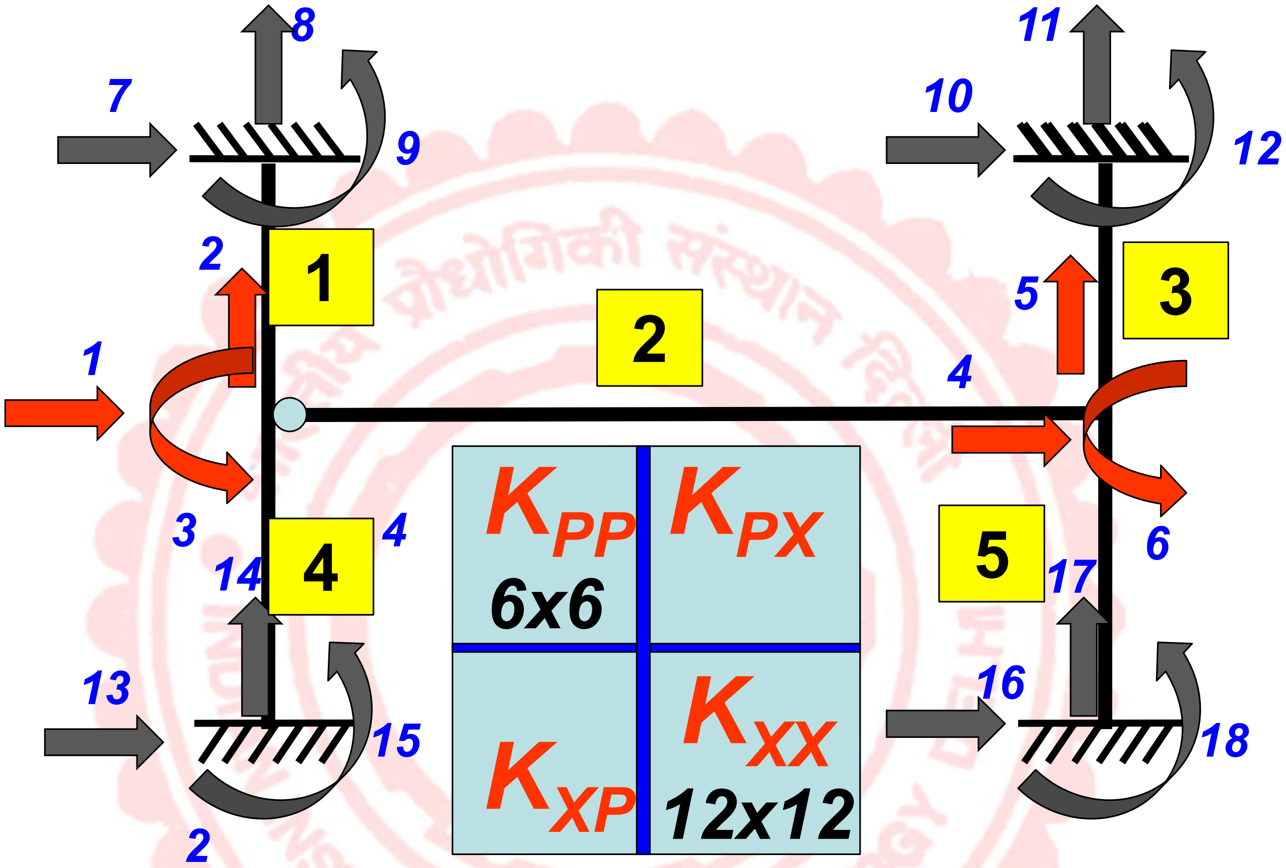
3rd row and 3rd column: All terms zero

Let us derive second column

ALTERNATE APPROACH: INTERNAL HINGE



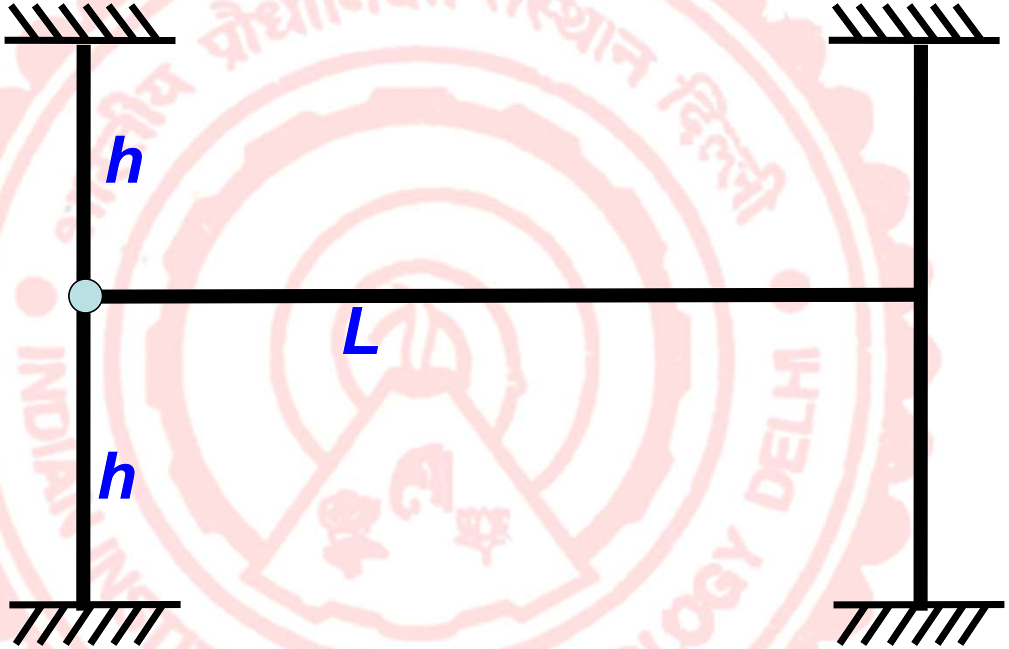
Use the code number approach, but here there is no duplication of the DOF as in the earlier approach.



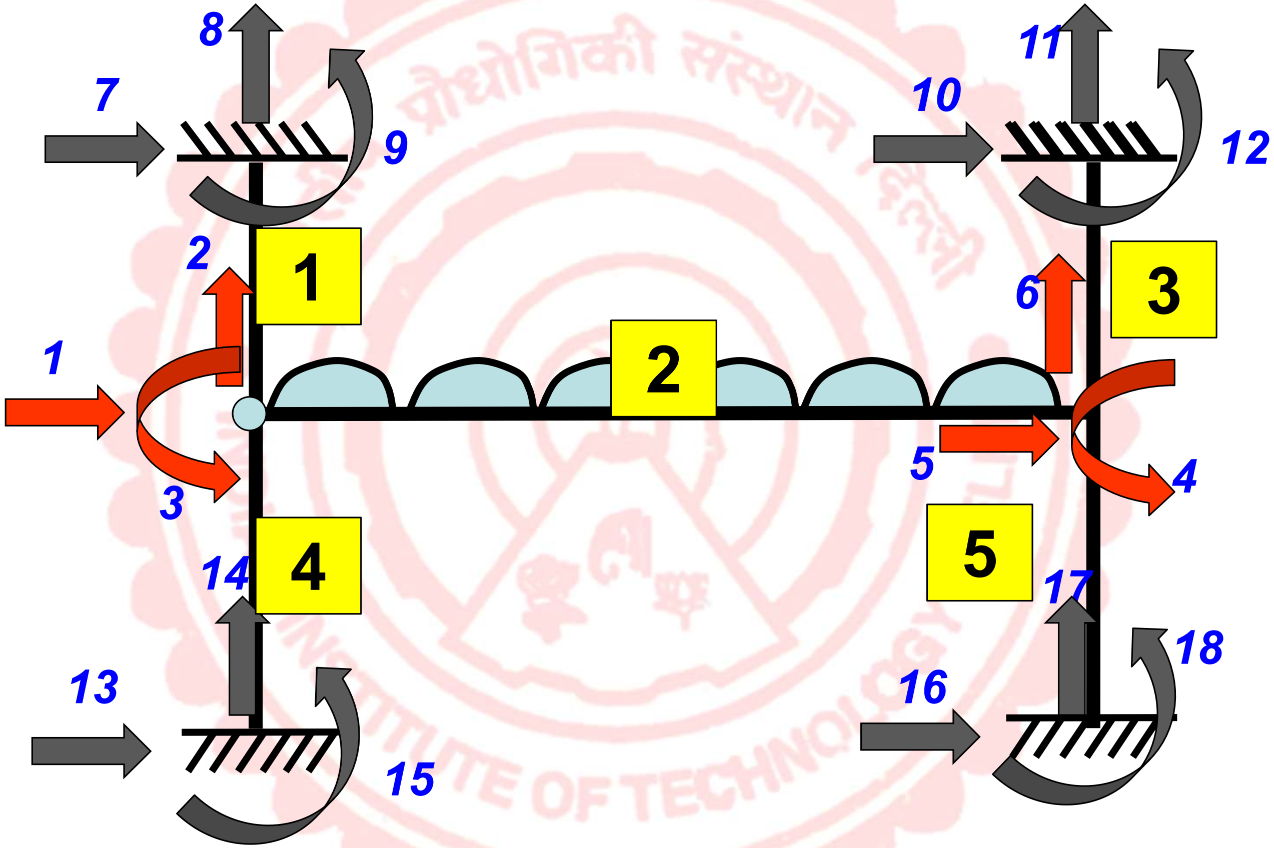
Automatically, member 2 will not make any contribution in the third row or column of K_{TS}. Members 1 and 4 will make contribution as before

The displacement corresponding to DOF 3 remains unknown for member 2. Corresponding displacement of the column can be obtained

ALTERNATE APPROACH: HINGE THROUGH BEAM AND COLUMNS BOTH



ALTERNATE APPROACH: HINGE THROUGH BEAM AND COLUMNS BOTH



No duplication of DOF as before……

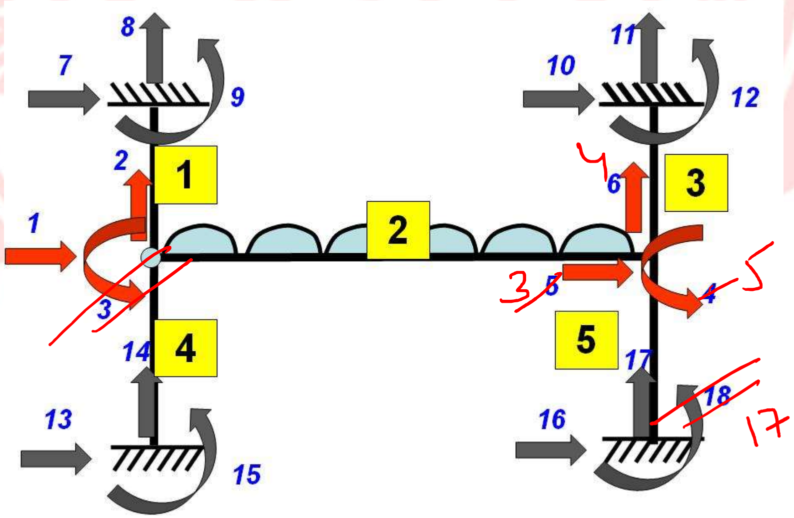
IMPLICATIONS

All three members 1, 2 and ~~3~~ 4 have modified $[K]_L$

After K_{TS} is formed, we will find the third row and third column to be zero. WHY????

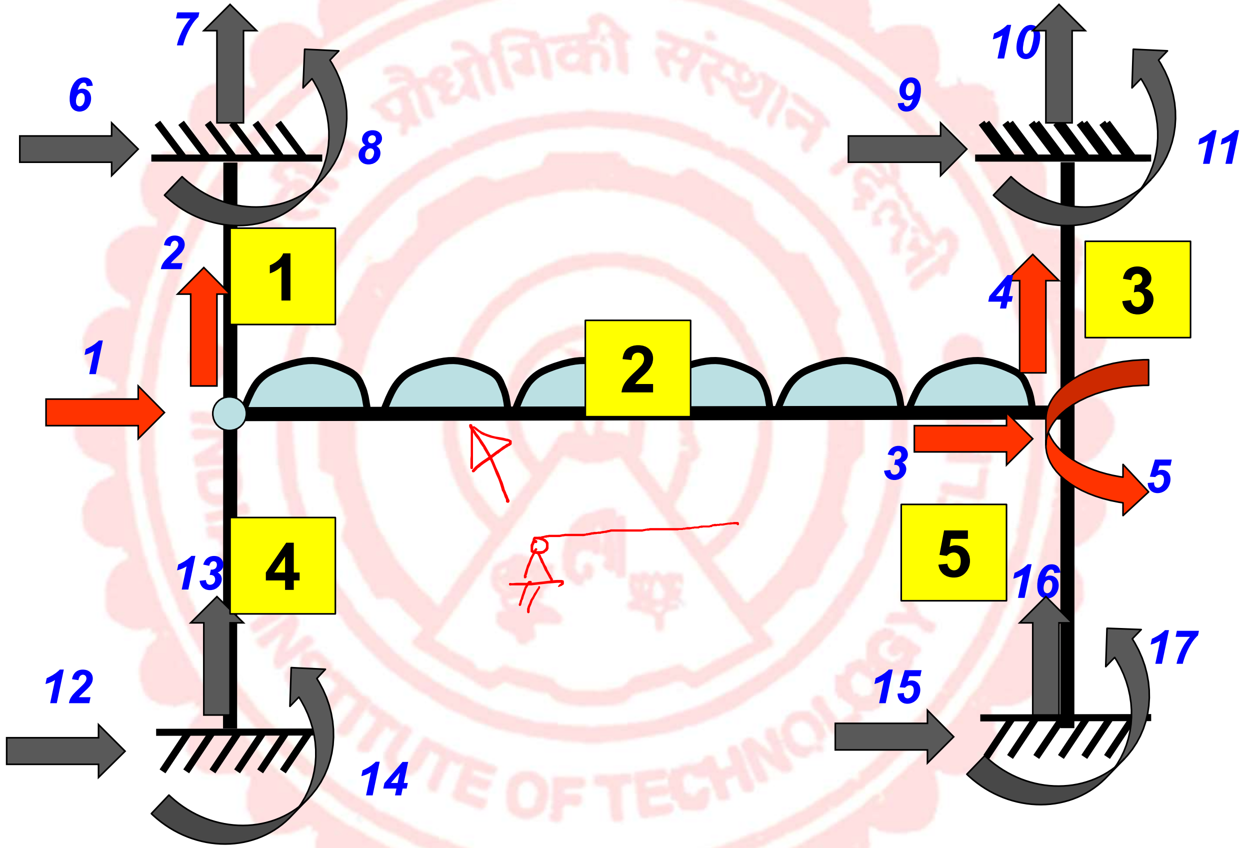
The diagonal element of K_{TS} (3,3) shall be ZERO. This would imply the matrix to be singular, $|K_{TS}| = 0$, hence, we will encounter run time error.

To circumvent this situation, eliminate the DOF (3). Renumber the DOFs and skip numbering this DOF.



$K_{TS} \rightarrow 17 \times 17$

ALTERNATE APPROACH: HINGE THROUGH BEAM AND COLUMNS BOTH

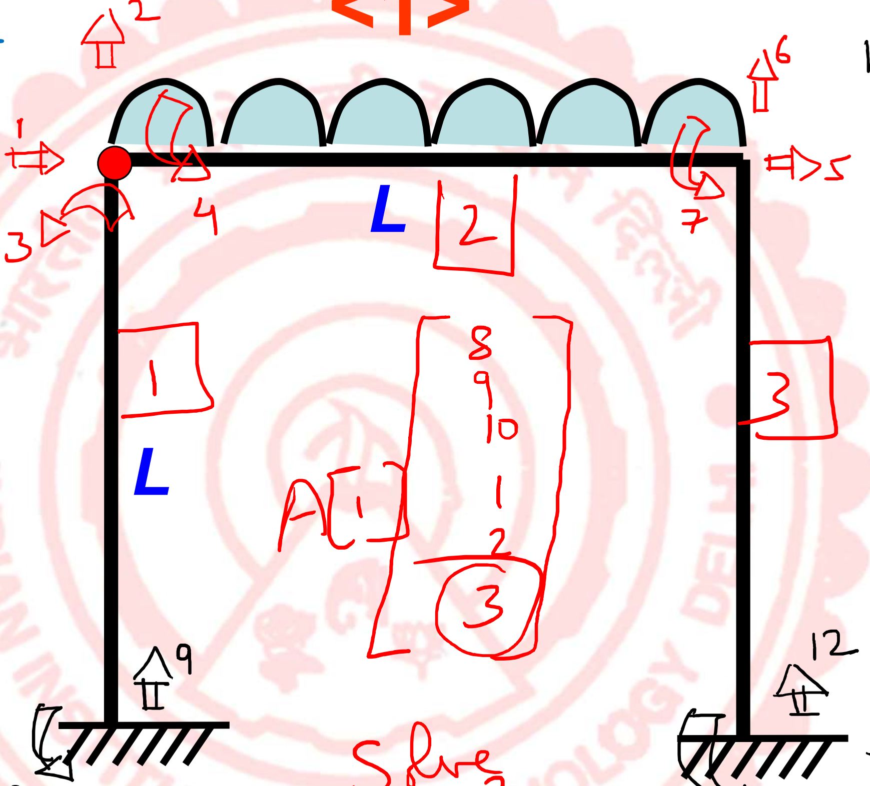


Need to skip the DOF corresponding to rotation (displacement output will be devoid of the values of these)

PRACTICE EXERCISE

<1>

With the duplication



K_{TS}: K_{PP} 7×7, 6×6; 13×13

$A[1] = [8, 9, 10, 1, 2, 3]$

$A[2] = [1, 2, 4, 5, 6, 7]$

Solve $\{u\}$

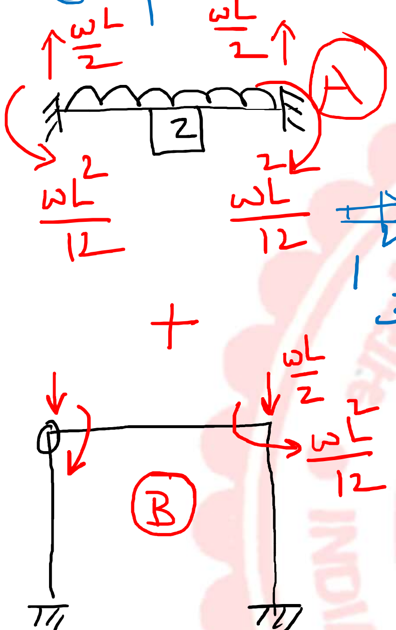
$$\{f\} + \{f\}$$

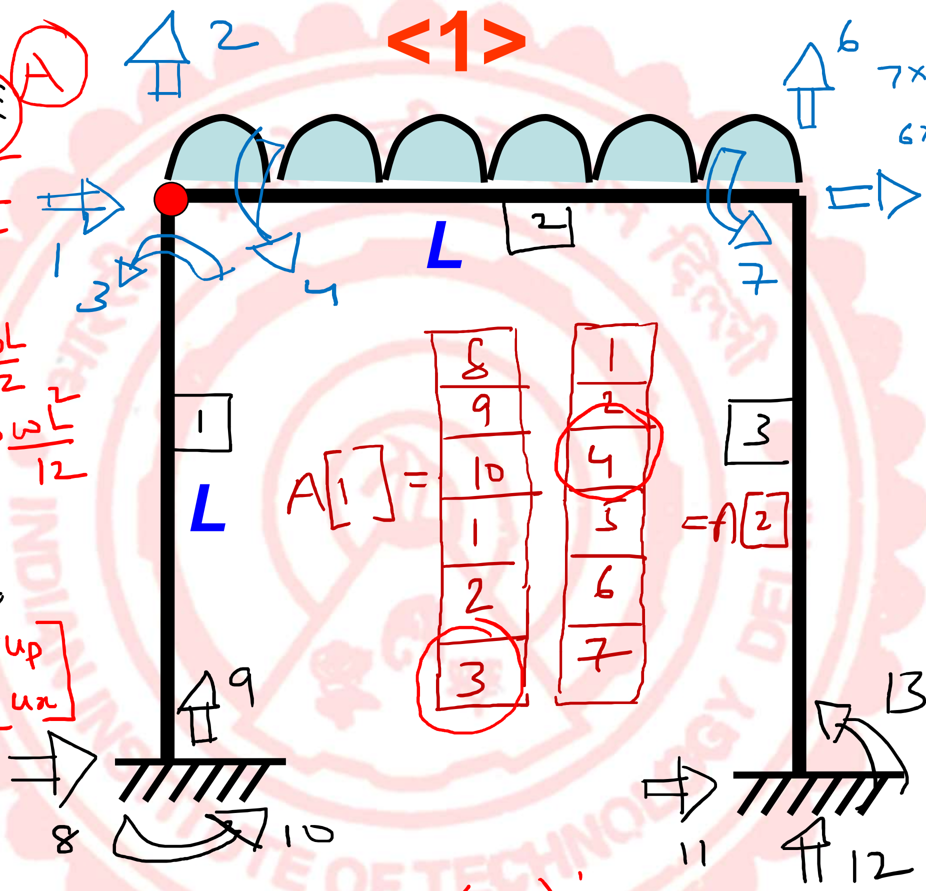
$$\begin{Bmatrix} P \\ X \end{Bmatrix} = \begin{bmatrix} 0 \\ -\frac{3}{8}wL \\ 0 \\ 0 \\ 0 \\ -\frac{5}{8}wL \\ \frac{wL^2}{8} \\ X_8 \\ X_9 \\ \vdots \\ X_{13} \end{bmatrix}$$

PRACTICE EXERCISE

<1>

Duplication





$$A[1] = \begin{bmatrix} 8 \\ 9 \\ 10 \\ 1 \\ 2 \\ 3 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 2 \\ 4 \\ 5 \\ 6 \\ 7 \end{bmatrix} = A[2]$$

$$\begin{array}{r} 7\times1 \\ 6\times1 \end{array}\begin{bmatrix} P \\ X \end{bmatrix} = \begin{array}{c} \uparrow \\ P \\ \downarrow \\ \uparrow \\ X \\ \downarrow \end{array} \begin{bmatrix} 0 \\ -wL/2 \\ 0 \\ -\frac{wL^2}{12} \\ 0 \\ -\frac{wL}{2} \\ +\frac{wL^2}{12} \\ \hline X_8 \\ X_9 \\ X_{10} \\ X_{11} \\ X_{12} \\ X_{13} \end{bmatrix} \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \end{array}$$

Solve

$$\begin{Bmatrix} P \\ X \end{Bmatrix} = \begin{bmatrix} K_{pp} & K_{px} \\ K_{xp} & K_{xx} \end{bmatrix} \begin{bmatrix} u_p \\ u_x \end{bmatrix}$$

$K_1^9(4,4) \longrightarrow$ Go to (1,1) in K_{TS}

Solve eqns $\{U_p\} \longrightarrow \{D\} \longrightarrow \{d\} = [T]\{D\} \longrightarrow \{f\} = [K_L]\{d\}$

Correction for Conc(A)

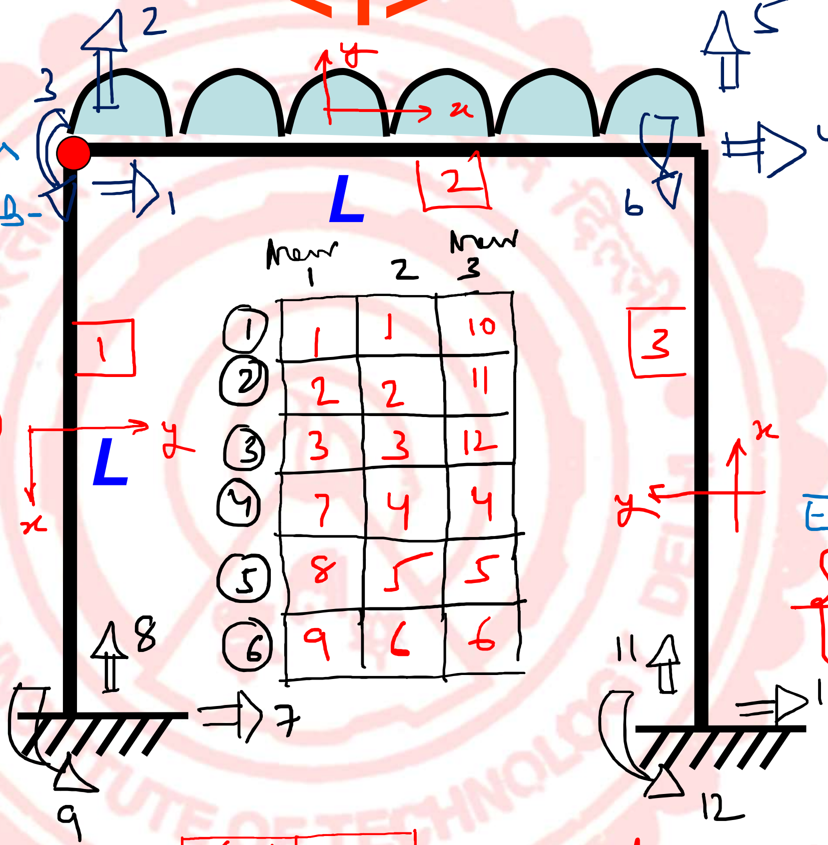
PRACTICE EXERCISE

<1>

Modify $[k]_L$

Mem 1, 2

Define local axes such that 1st memb-end joint is hinge



Mem 1	2	Mem 3
① 1	1	10
② 2	2	11
③ 3	3	12
④ 7	4	4
⑤ 8	5	5
⑥ 9	6	6

$K_L^{①*}$, $K_L^{②*}$, $K_L^{③}$

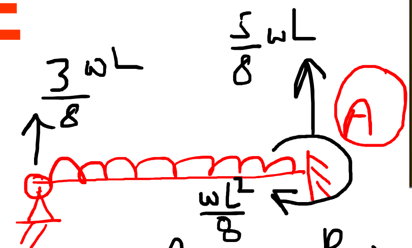
Modified

$[T]^T[K_L][T]$

$[K_G]$ for mems 1, 2, 3

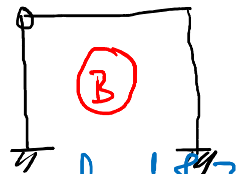
6x6 K_{pp}	
	6x6 K_{xx}

12 x 12 ⟹ 3rd row & 3rd col zero



Cant choose other option

+



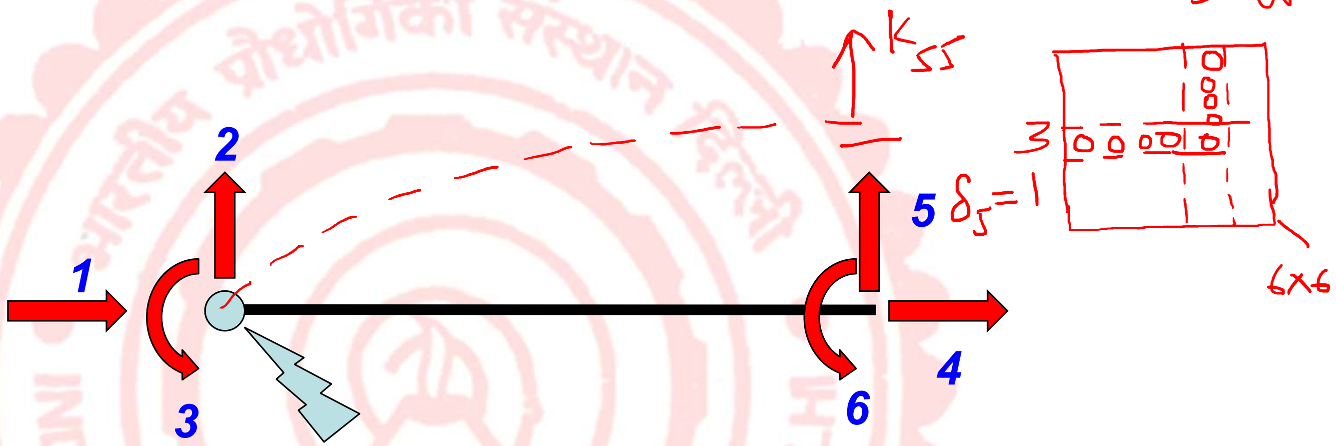
Eliminate dof 3

$$\begin{Bmatrix} P \\ X \end{Bmatrix} = \left[\begin{array}{c|c} K & \\ \hline & TS \end{array}\right] \begin{Bmatrix} u_p \\ u_x \end{Bmatrix}$$

11x1   11x11   11x1

We don't get any disp. corr. to dof $\boxed{3}$

INTERNAL HINGE : ALTERNATE APPROACH- TO MODIFY $[K]_L$



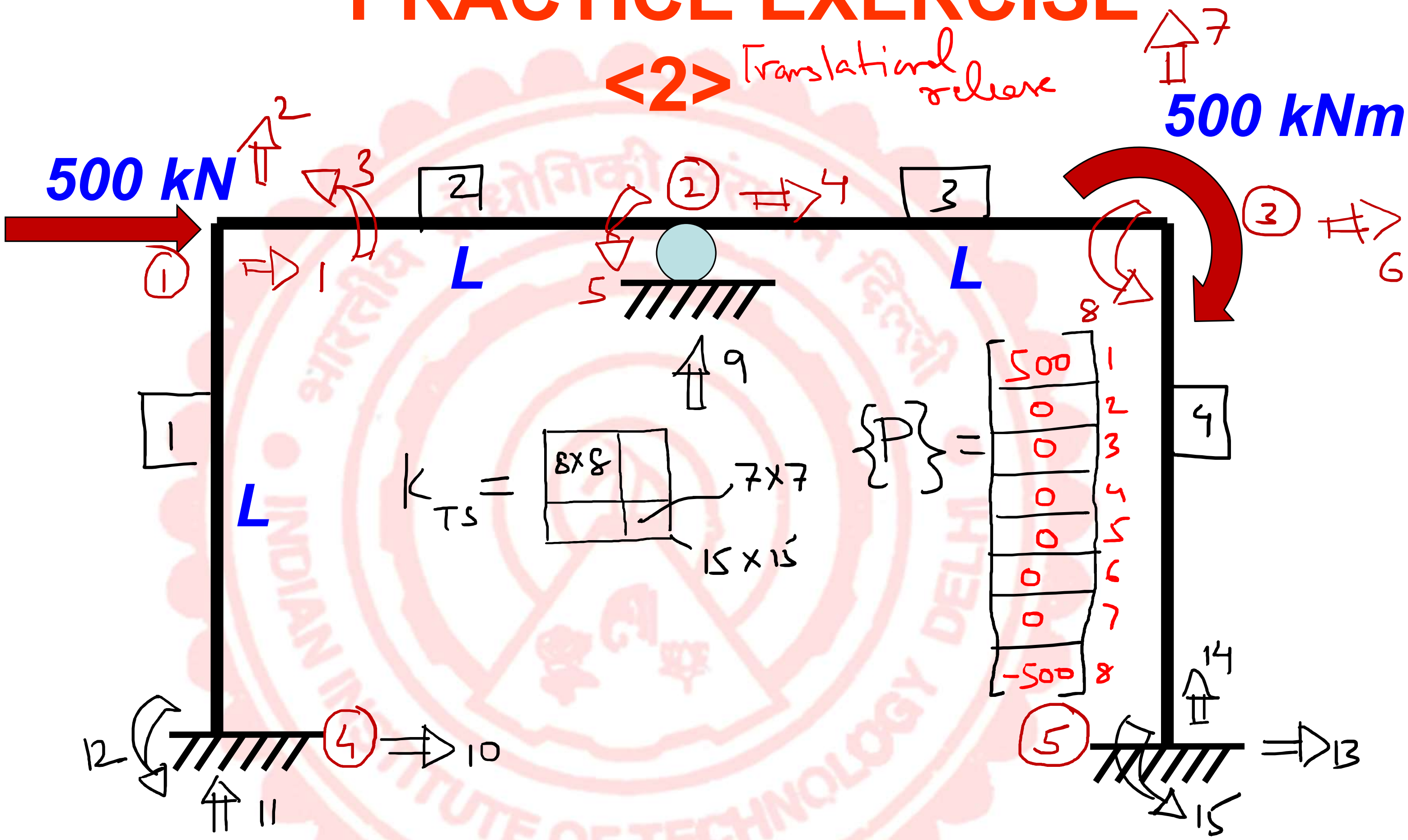
Permanent hinge

This boundary condition is not to be altered while deriving the member stiffness matrix

Hinge not to be fixed while deriving the member stiffness matrix

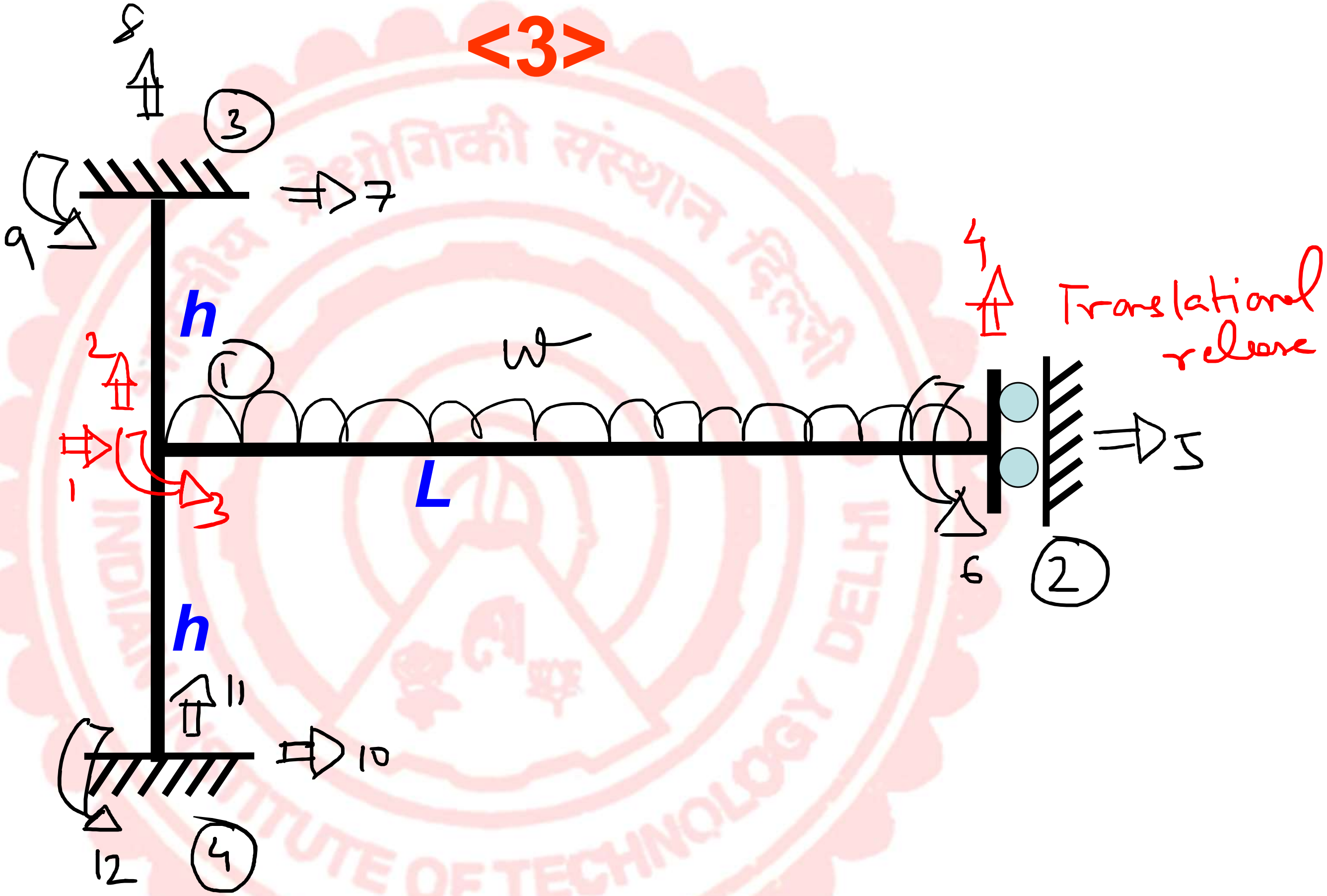
PRACTICE EXERCISE

<2>



PRACTICE EXERCISE

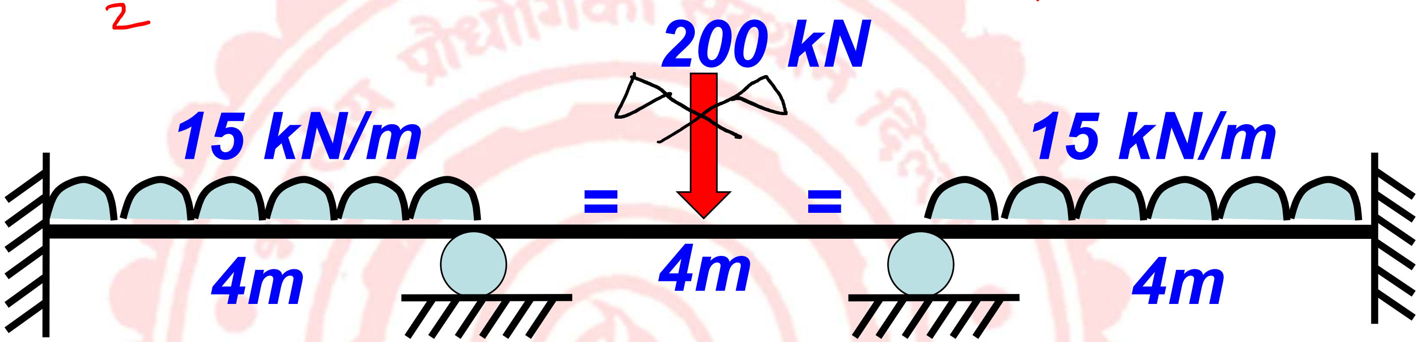
<3>

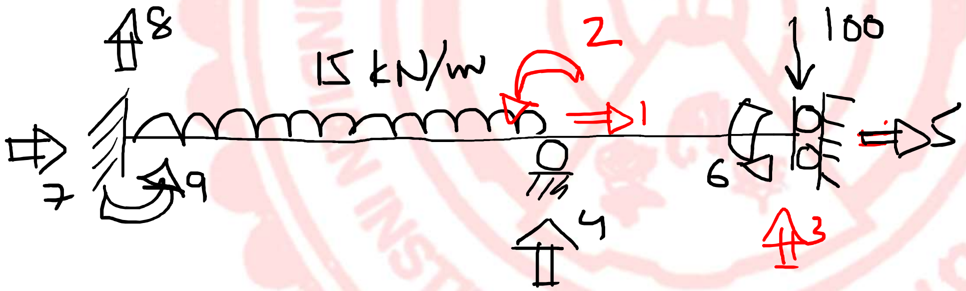


PRACTICE EXERCISE

<4>

$$\frac{15\times 4}{2} = 30\,kN \qquad \frac{15\times 4^2}{12} = 20\,kNm$$

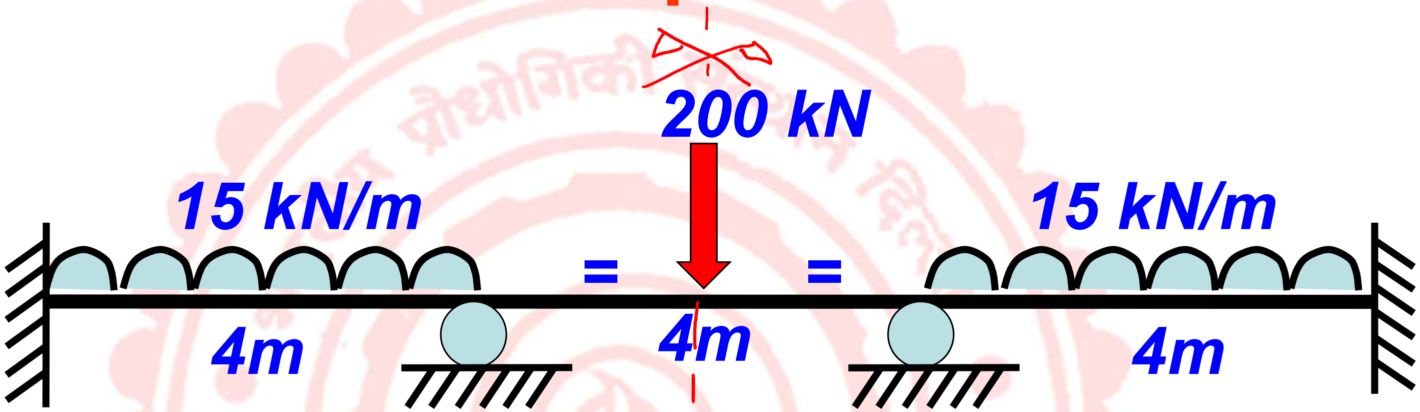


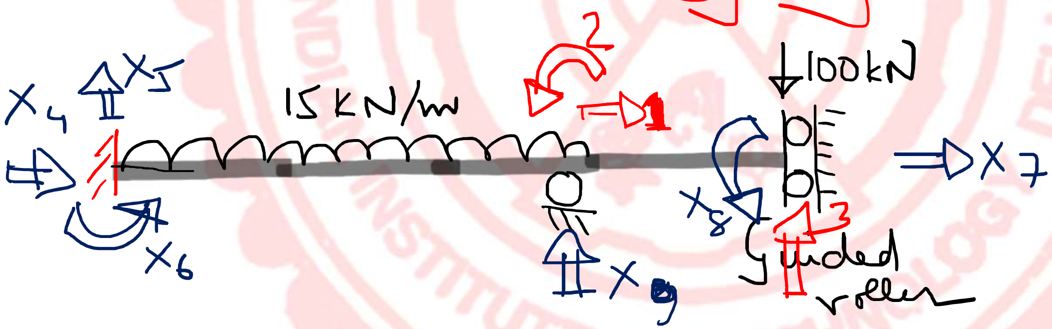


$$\begin{Bmatrix} P_{3\times1} \\ \hline X_{6\times1} \end{Bmatrix} = \begin{array}{|c|c} 0 & 1 \\ 20 & 2 \\ -100 & 3 \\ X_4-30 & 4 \\ X_5 & 5 \\ X_6 & 6 \\ & 7 \\ X_8-30 & 8 \\ X_9-20 & 9 \end{array}$$

PRACTICE EXERCISE

<4>

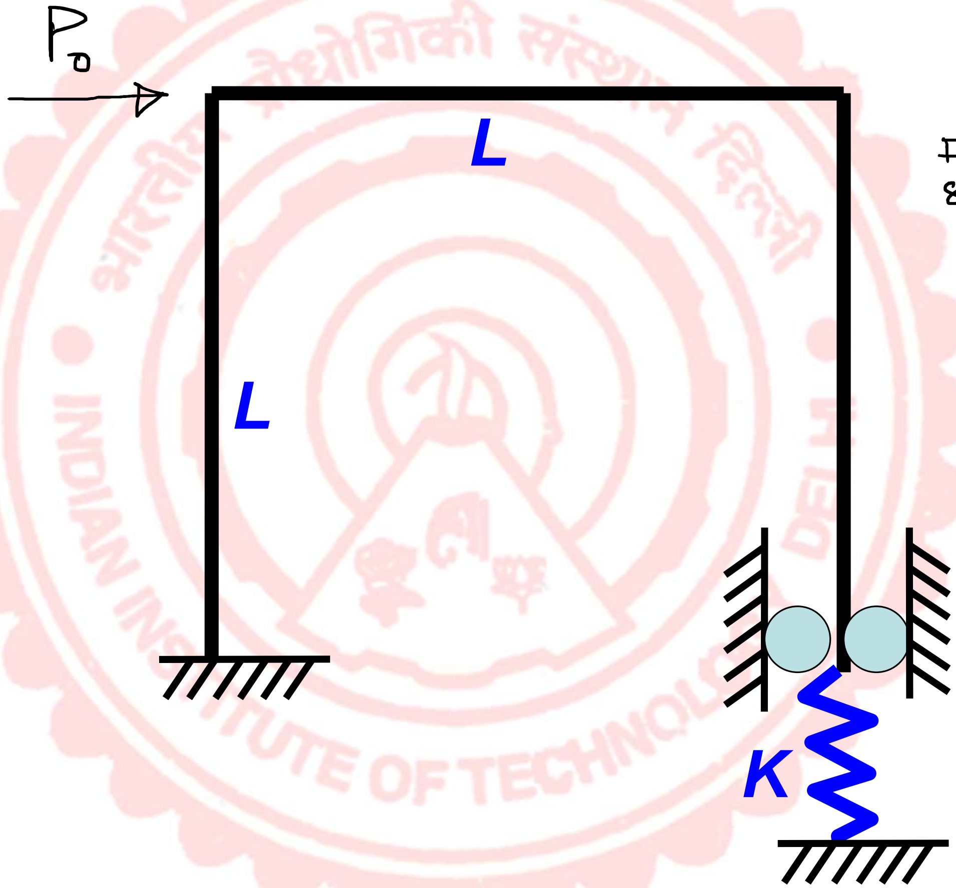


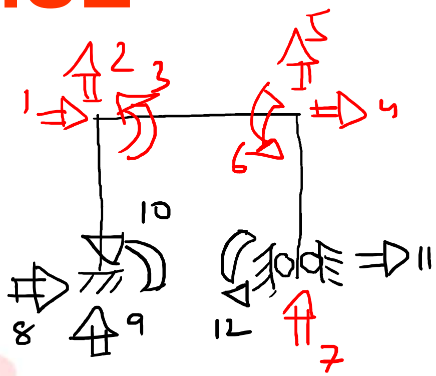


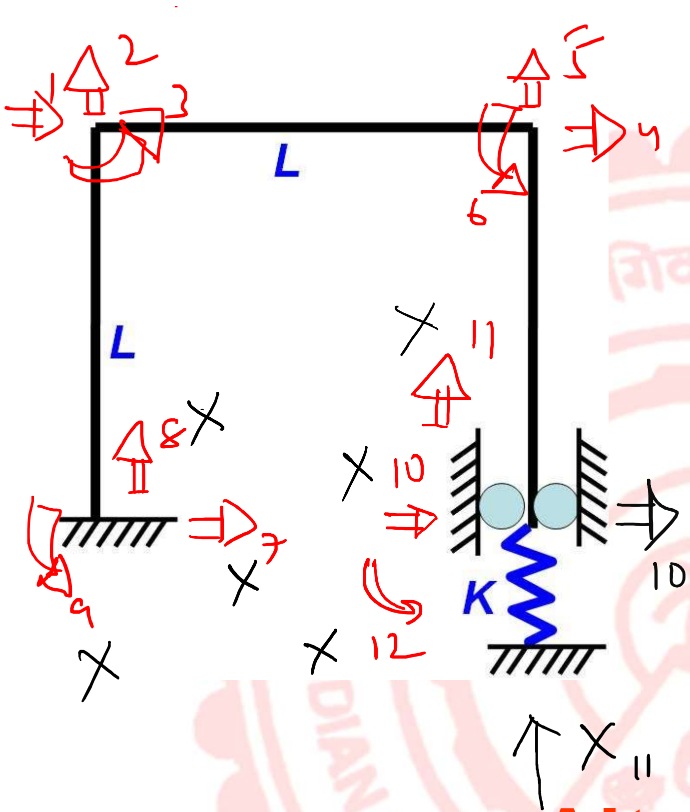
$P_3 = -100\text{ kN}$

PRACTICE EXERCISE

<5>







$$\begin{Bmatrix} \underset{6\times1}{P} \\ \underset{6\times1}{X} \end{Bmatrix} = \begin{bmatrix} \underset{6\times6}{K_{PP}} & \\ & \underset{6\times6}{K_{XX}} \end{bmatrix} \begin{Bmatrix} \underset{6\times1}{u_p} \\ \underset{6\times1}{u_x} \end{Bmatrix}$$

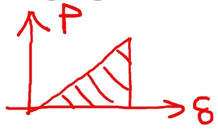
$$u_{11} = \frac{-X_{11}}{k}$$

Alternate approach:

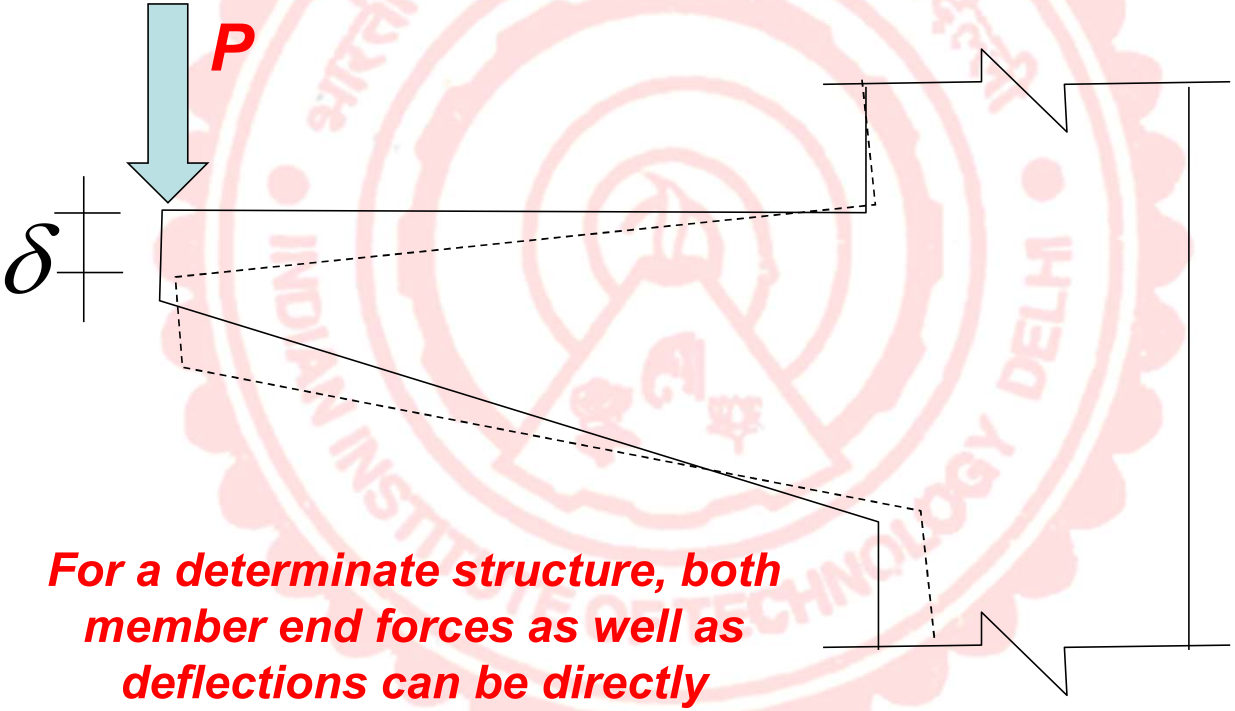
Add one more member (a link element)

TREATMENT OF NON-PRISMATIC MEMEBERS

Case I: Determinate Structures

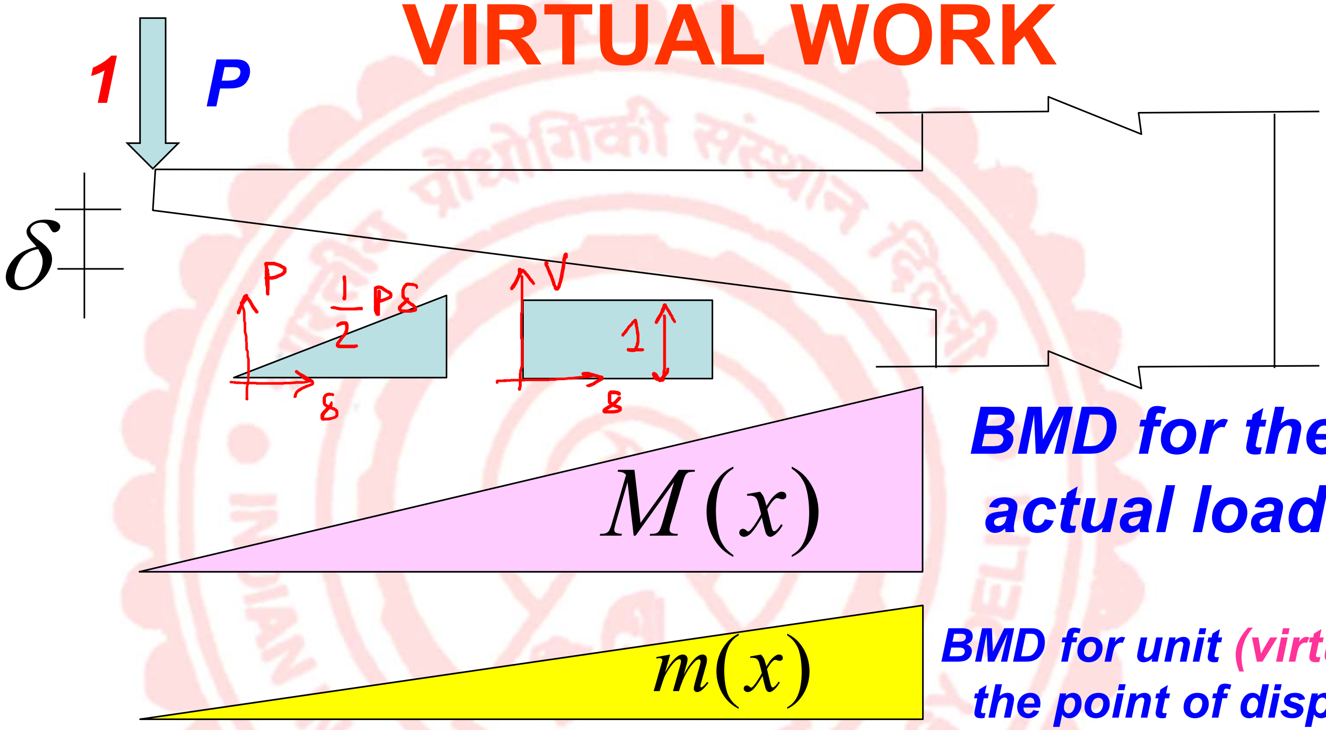


$\frac{1}{2} P\delta = W$



For a determinate structure, both member end forces as well as deflections can be directly calculated…..

SOLUTION BY PRINICIPLE OF VIRTUAL WORK



BMD for the actual load

BMD for unit (virtual) load at the point of displacement

Int. Virtual Work = Ext. Virtual Work

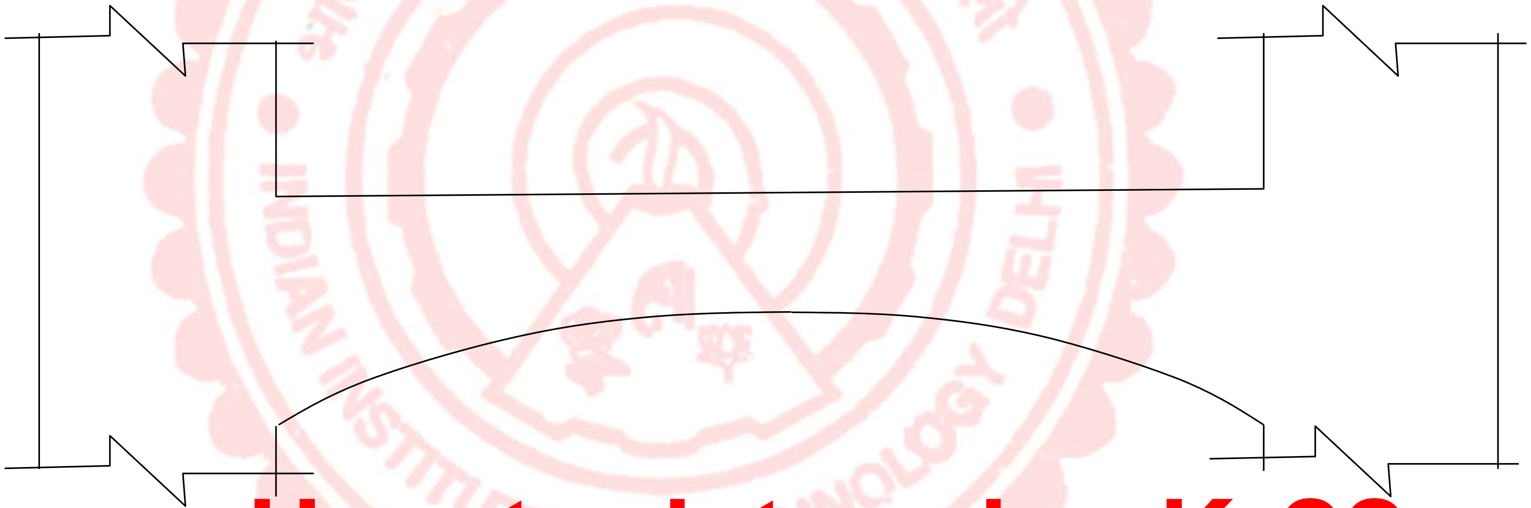
$$1.\delta = \int \frac{M(x)m(x)}{EI}dx$$

To take care of the non-prismatic nature of the member

Conclusion: ***For determinate structures, both member end forces and deflections can be easily computed by incorporating the variation of EI***

TREATMENT OF NON-PRISMATIC MEMEBERS

Case II: Indeterminate Structures



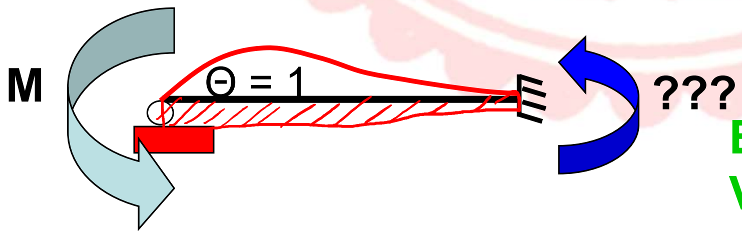
How to determine K_L??

Any issue?

TREATMENT OF NON-PRISMATIC MEMEBERS

Case II: Indeterminate Structures

In order to derive stiffness matrix, as per first principles, we need to apply unit displacement along a particular DOF keeping all other displacements zero

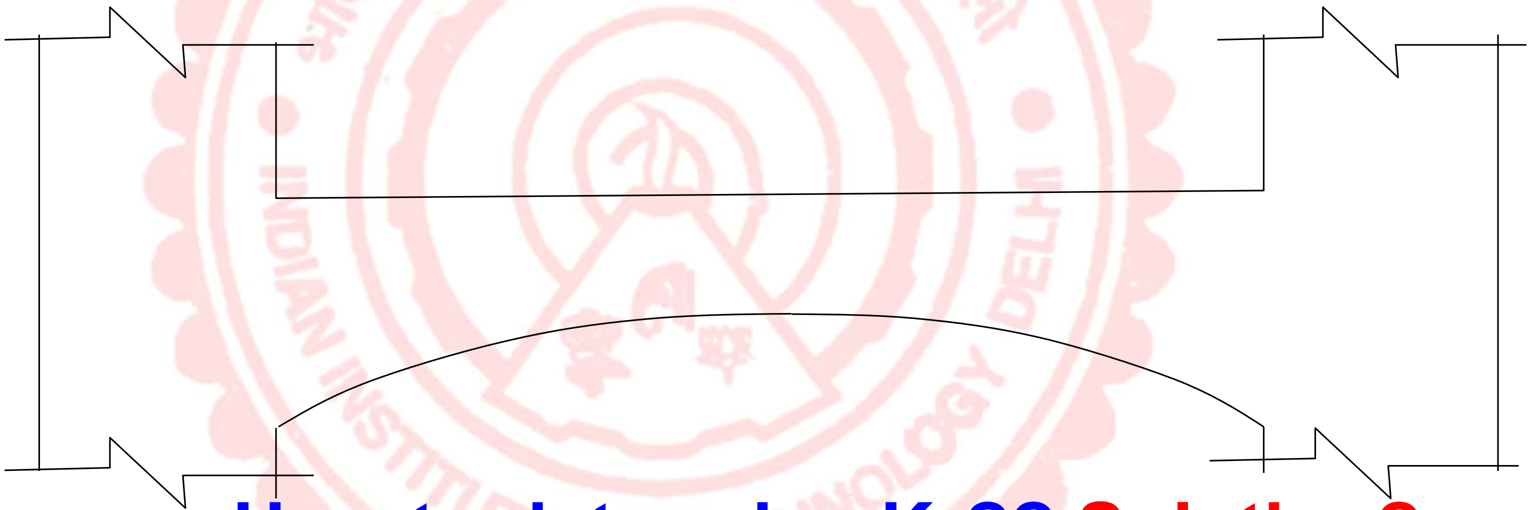


Basic slope-deflection formulations no longer valid. Use of classical force method too tedious..

Being indeterminate, principle of Virtual Work cannot be applied

TREATMENT OF NON-PRISMATIC MEMEBERS

Case II: Indeterminate Structures



How to determine K_L?? Solution?

We have to use indirect approach, employing flexibility method

WE WILL UTILIZE THE FLEXIBILITY METHOD

F_{ij} =

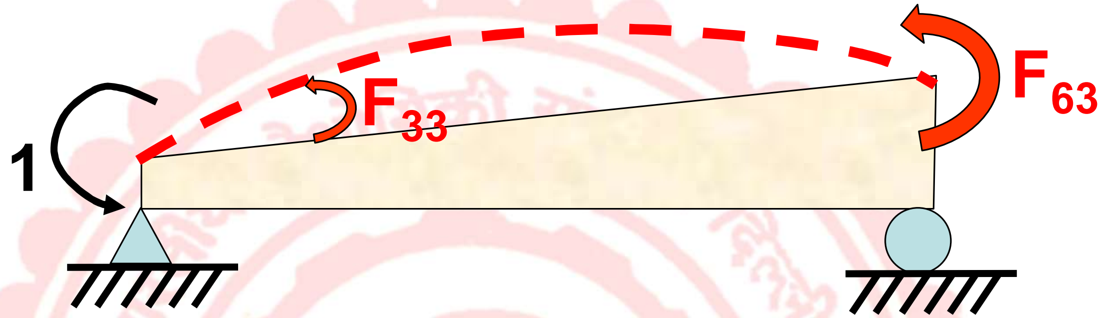
Displacement along the line of action of the i^{th} force when we apply unit force along the line of action of the j^{th} force….such that…

(no force acting along the lines of action of other designated forces=> no restraint)

Unlike the stiffness approach (which emphasizes on locking remaining displacements), this process creates a determinate structure…

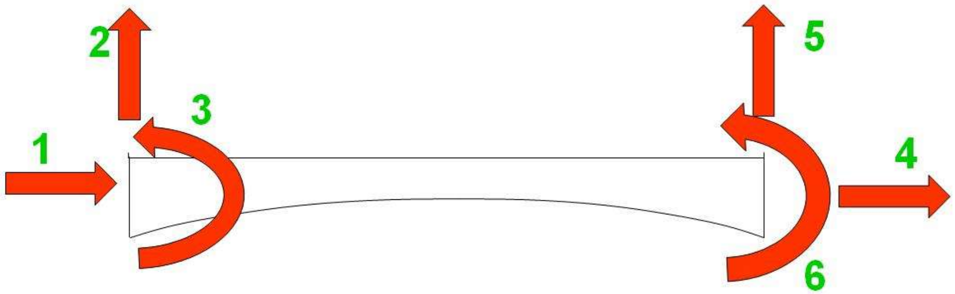
How this is done?....see the next step.

USE OF FLEXIBILITY METHOD

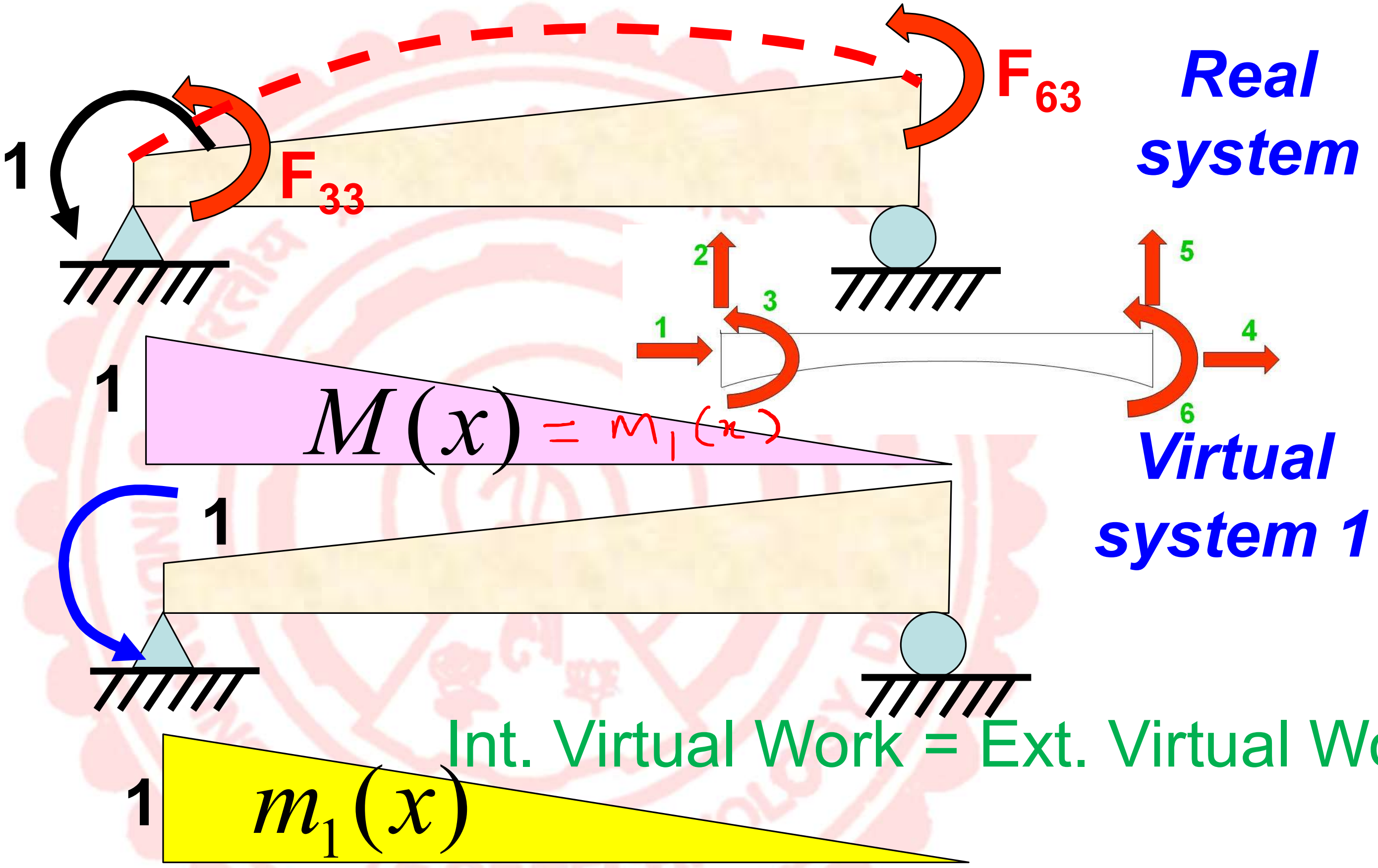


Apply unit force along "3", no force to be applied along other force lines.

Hence, no other force is generated, except reactions. The structure is determinate, so that we may easily apply the principle of virtual work



APPLY PRINCIPLE OF VIRTUAL WORK

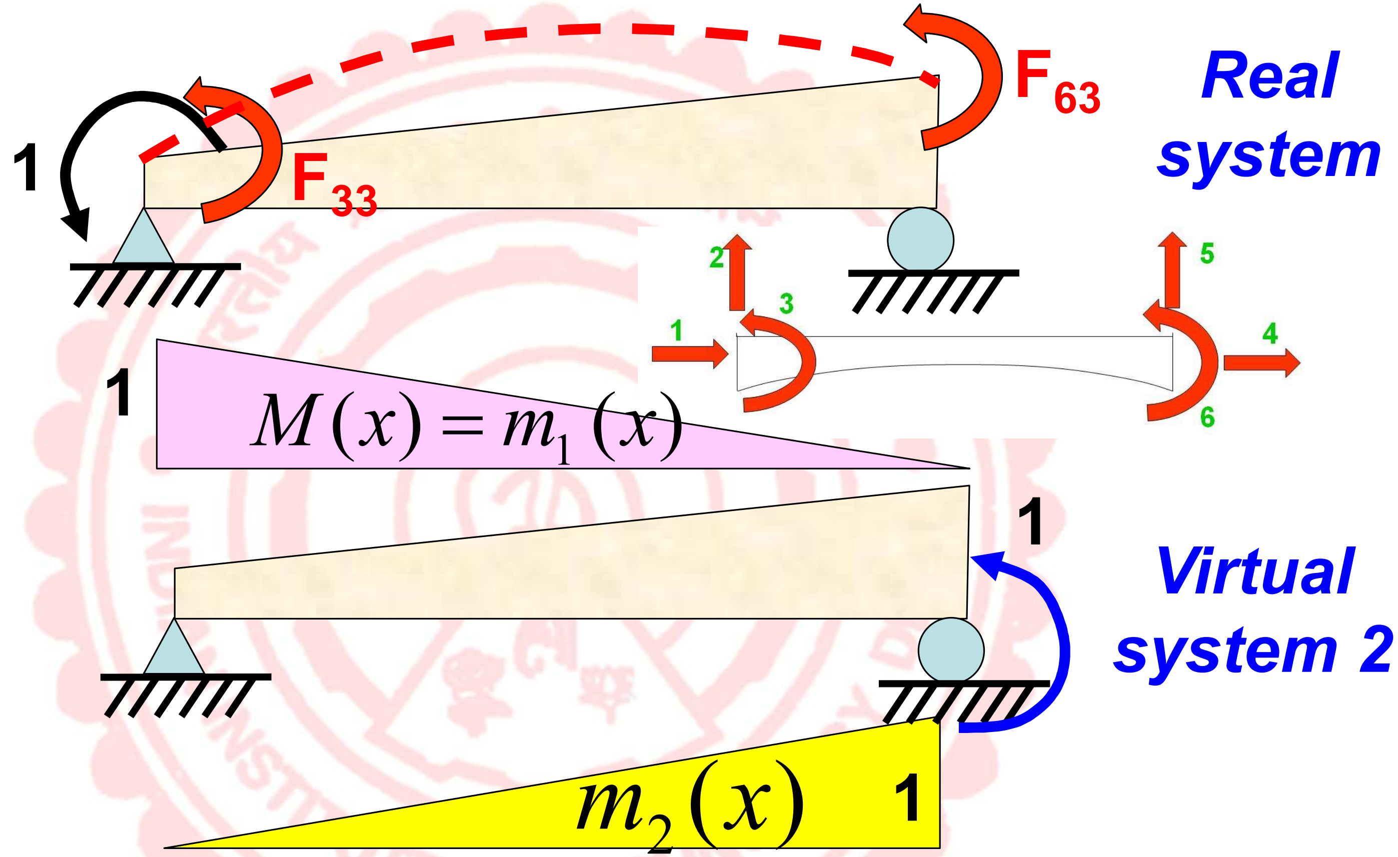


Int. Virtual Work = Ext. Virtual Work

$$1.F_{33} = \int_0^L \frac{M(x)m(x)}{EI}dx$$

$$F_{33} = \int_0^L \frac{[m_1(x)]^2}{EI}dx$$

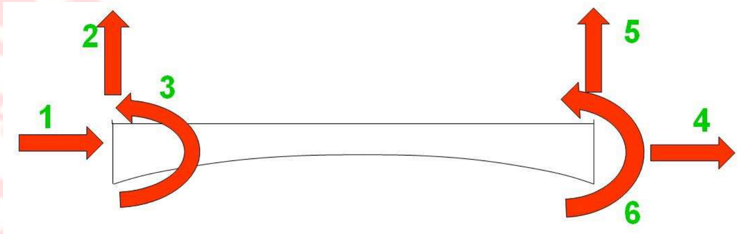
APPLY PRINCIPLE OF VIRTUAL WORK

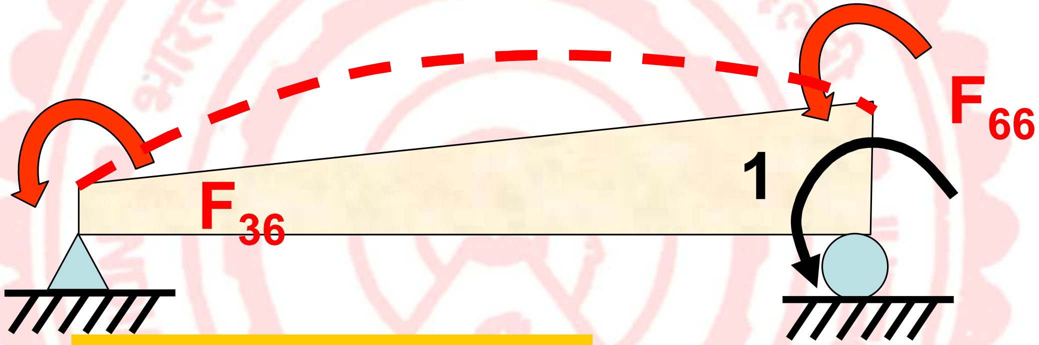


$$1.F_{63} = \int_0^L \frac{M(x)m(x)}{EI}dx$$

$$F_{63} = \int_0^L \frac{m_1(x)m_2(x)}{EI}dx$$

SIMILARLY….





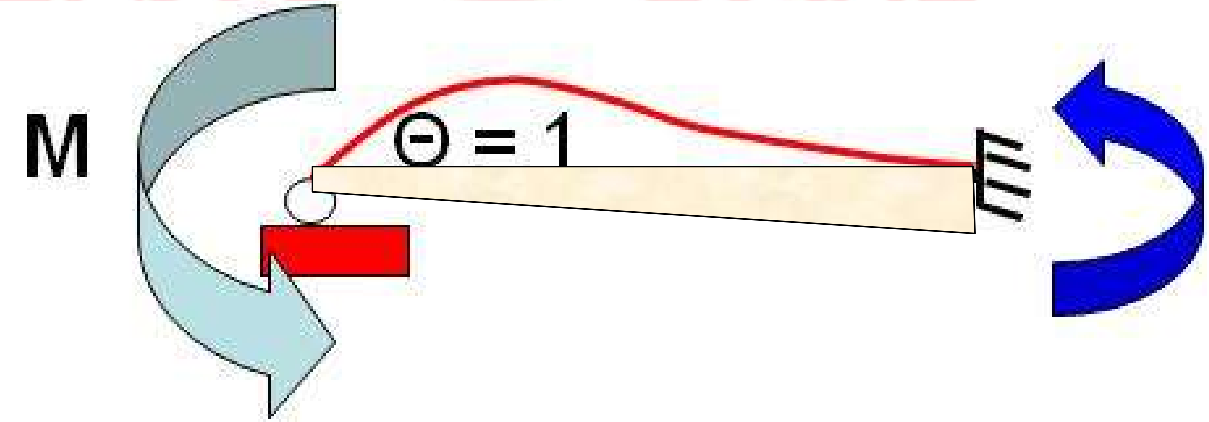
Real system

$$F_{66} = \int_0^L \frac{[m_2(x)]^2}{EI} dx$$

$$F_{36} = F_{63}$$

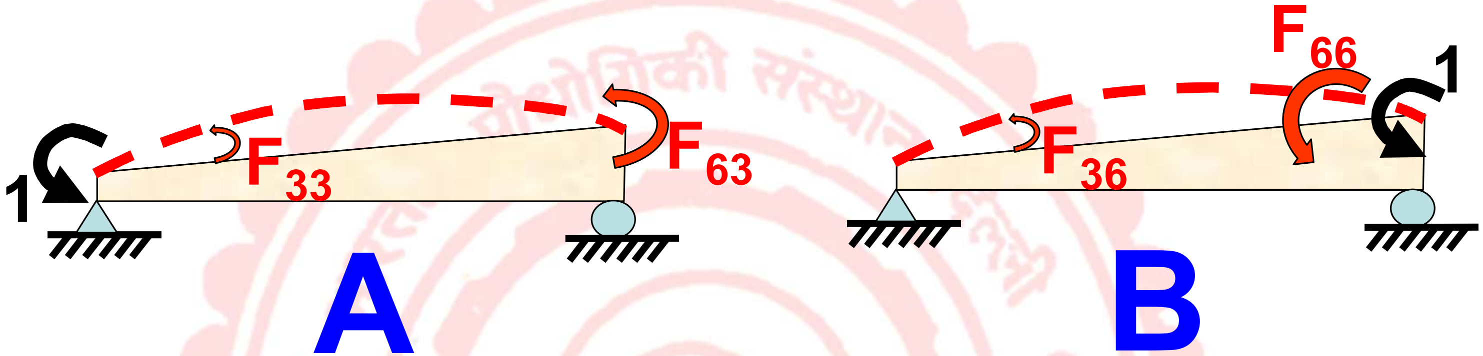
$$\int_0^L \frac{m_1(x) m_2(x)}{EI} dx$$

BUT THIS IS NOT WHAT WE FINALLY WANT



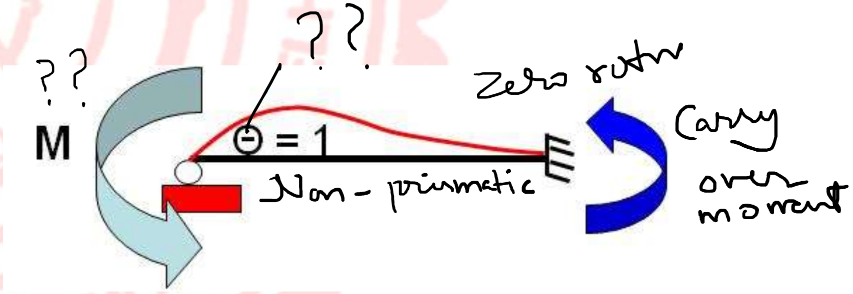
Can we use superposition??

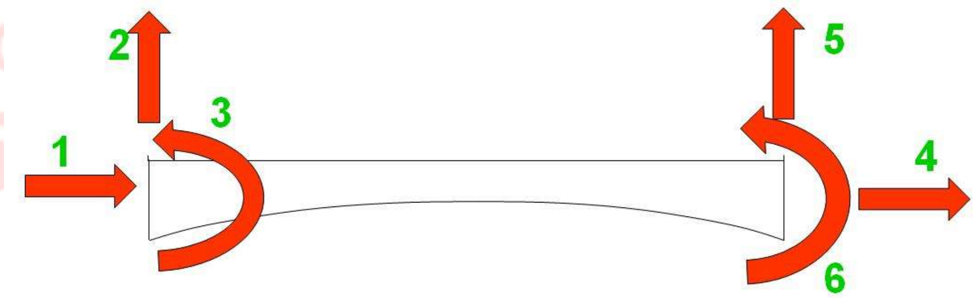
WE USE THE PRINCIPLE OF SUPERPOSITION TO GET FINAL SOLUTION

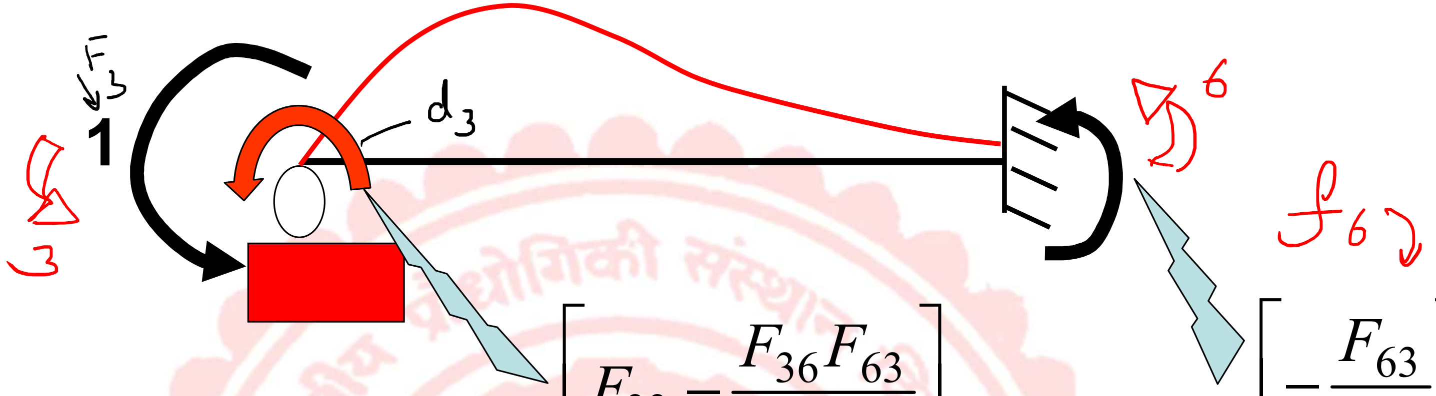


Combine (A) and (B) in following fashion:

$$A - \left[\frac{F_{63}}{F_{66}}\right] B$$







$$\underbrace{\left[F_{33}-\frac{F_{36}F_{63}}{F_{66}}\right]}_{d_3} \qquad \left[-\frac{F_{63}}{F_{66}}\right]$$

$$K_{63}=\frac{f_6}{d_3}$$

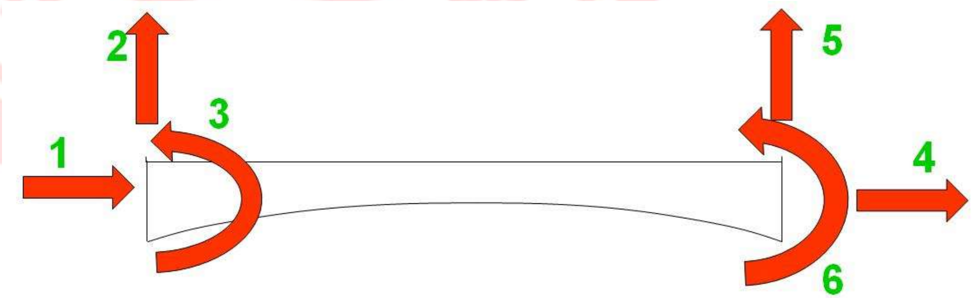
Hence, by definition

$$K_{33}=\frac{1}{\left[F_{33}-\frac{F_{36}F_{63}}{F_{66}}\right]}=\frac{F_{66}}{F_{33}F_{66}-F_{36}^{2}}$$

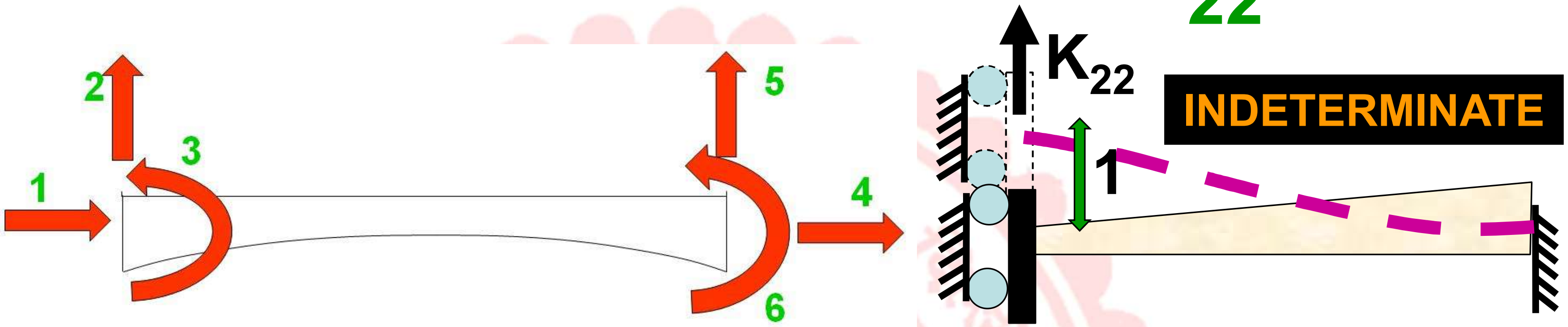
Similarly,

$$K_{66} = \frac{F_{33}}{F_{33}F_{66} - F_{36}^2}$$

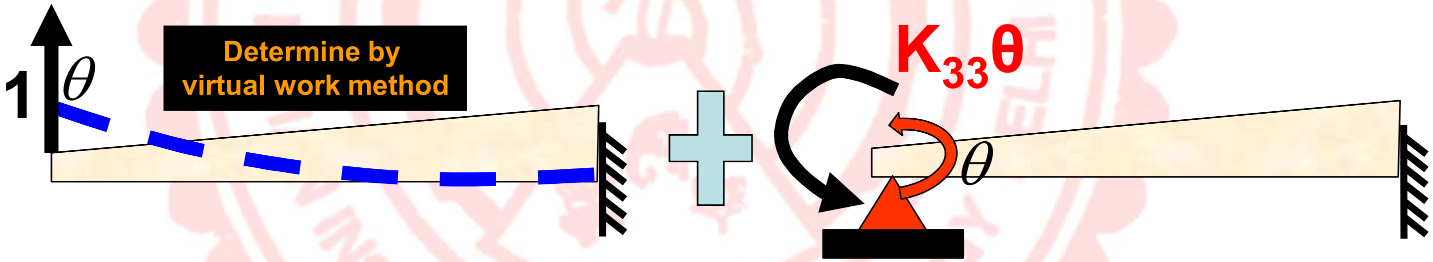
$$K_{63} = \frac{-F_{63}}{F_{33}F_{66} - F_{36}^2}$$



HOW TO DERIVE K_{22}



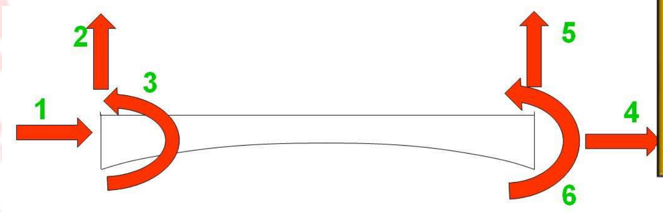
CONSIDER COMBINATION OF

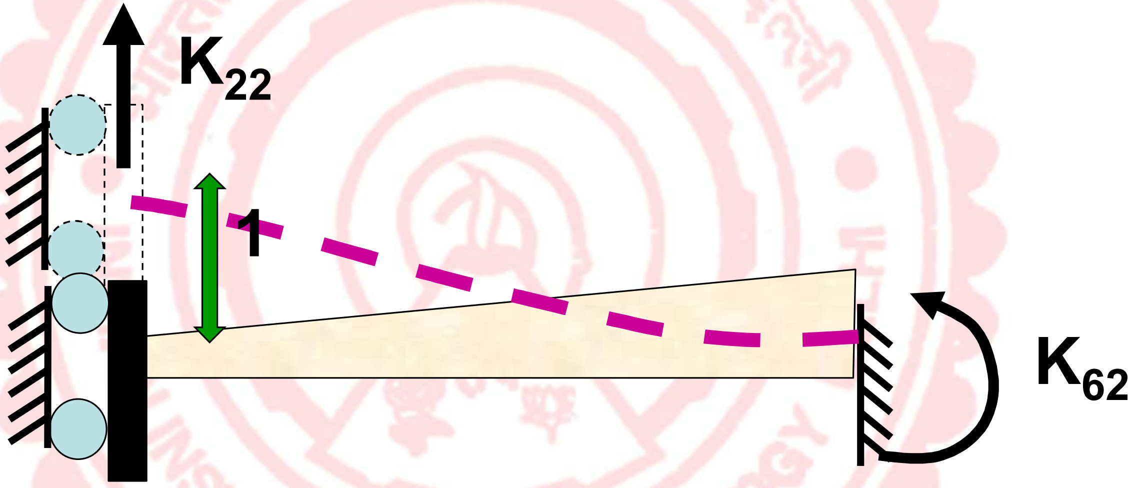


A **B**

Choose multipliers such that the net angle of rotation on the left end of the beam is ZERO

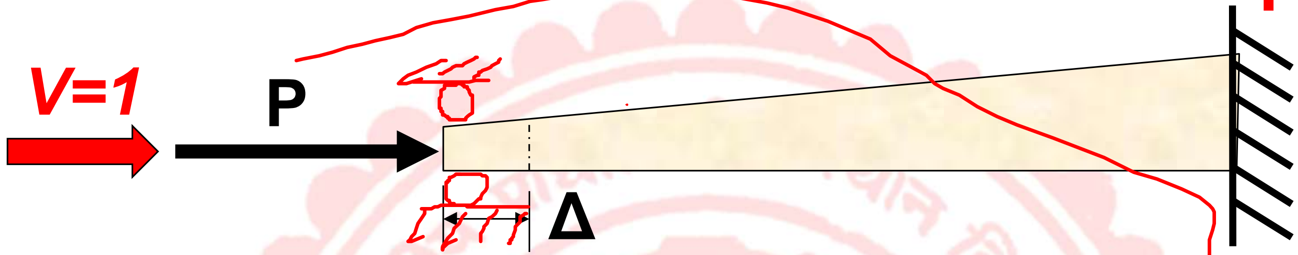
OTHER ELEMENTS OF $[K]_L$





Similarly derive K_{55}, K_{35}

HOW TO DERIVE K_{11}



Apply the Principle of Virtual Work

External Virtual Work = External Virtual Work

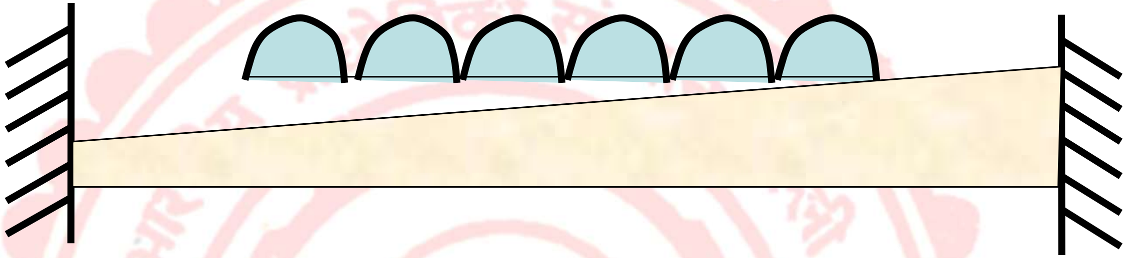
$$1.\Delta = \int_0^L F_V \varepsilon_R \, dx$$

From actual loads

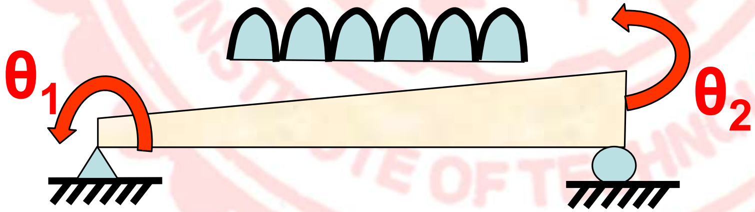
$$= \int_0^L F_V \left(\frac{F_R}{AE} \right) dx$$

NON-PRISMATIC MEMEBERS

How to obtain fixed ended forces?

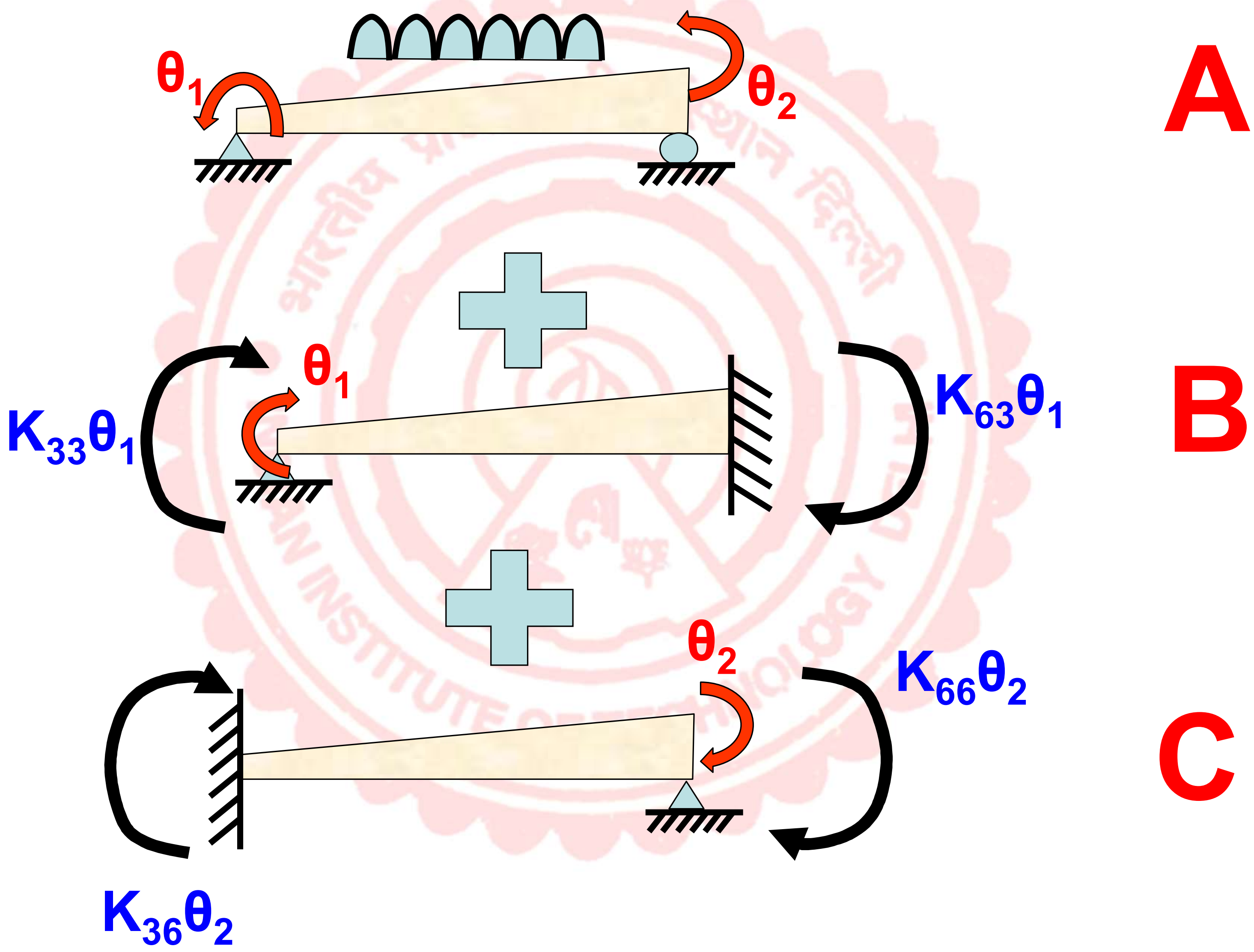


Release the restraints and convert the structure into a determinate system

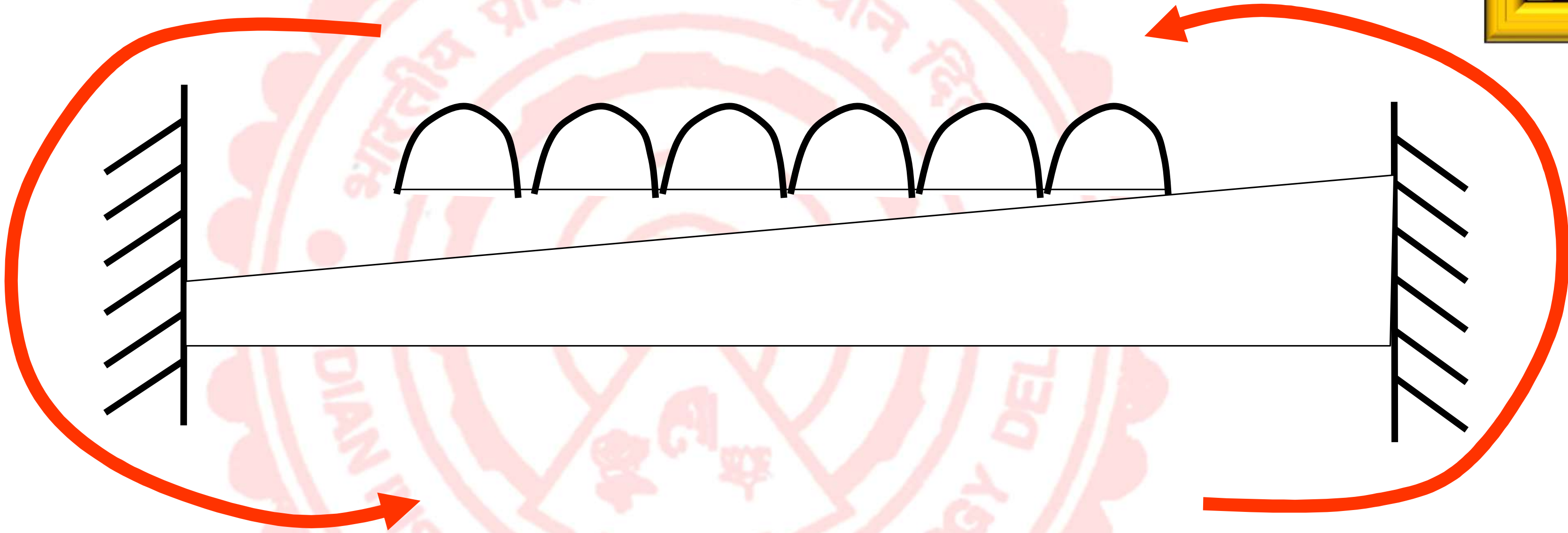


Both rotations can be obtained using the principle of virtual work

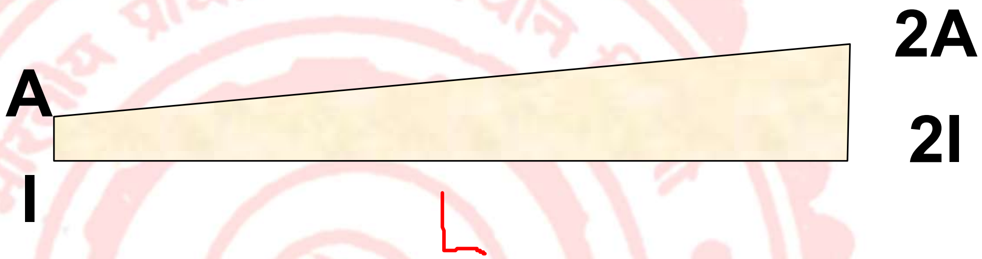
Superimpose the A, B, C:



A+B+C



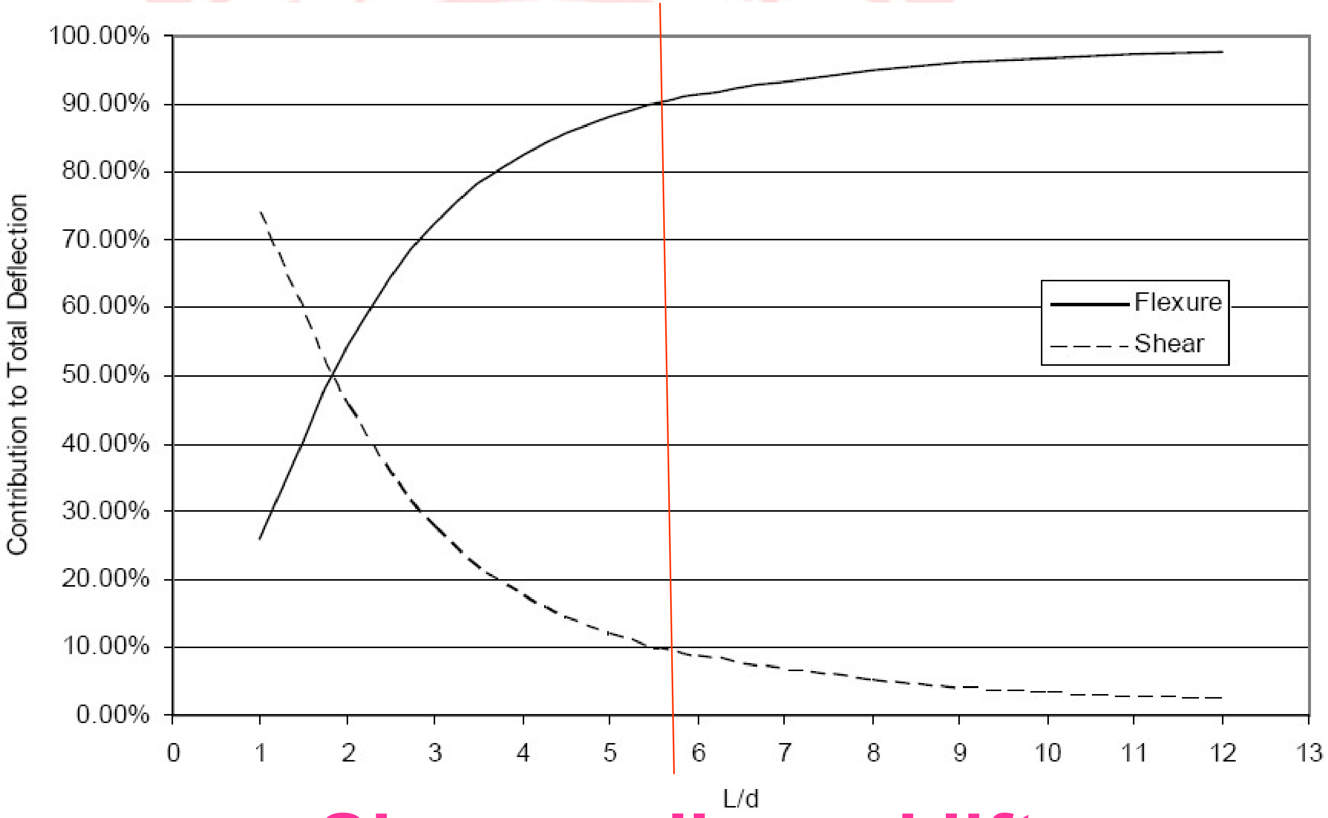
PRACTICE EXERCISE



Derive the stiffness matrix

INCLUSION OF SHEAR DEFORMATION EFFECT

Necessary for deep sections L/D <= 6



Shear walls and lift cores

INCLUSION OF SHEAR DEFORMATION EFFECT

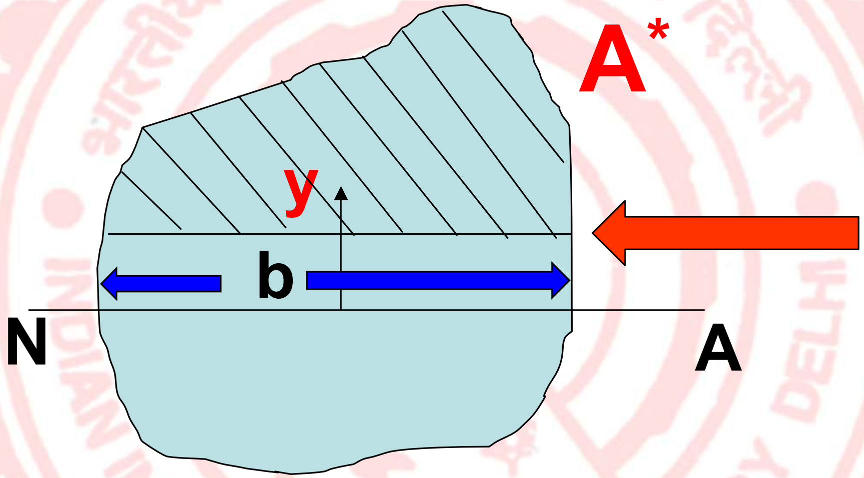
Treatment shall be restricted to prismatic sections only

In any deformed member, strain energy is given by

$$U = \iiint_V \frac{1}{2}\sigma\varepsilon dV$$

WHY FORM FACTOR

The shear stress varies across the height of the cross-section.



$$\tau = \frac{VQ}{Ib}$$

$$\boxed{Q(y) = \int_{A^*} y \, dA}$$

First moment of part of cross-section above the section considered

To simplify the computation, shear stress is assumed to be uniform across the cross section, which is strictly not correct.

Form factor is introduced to apply correction for non-uniform shear stress, such that equivalent uniform stress gives same results as with actual non-uniform shear shear stress.

$$\tau_{uniform} = s\left(\frac{V}{A}\right)$$

S> 1

$$\tau_{uniform} = \left(\frac{V}{A/s}\right) = \frac{V}{A_{eff}}$$

FORM FACTOR (DEF.)

Form factor is defined as the ratio of the gross area of the section to the shear area of the section

$$s = \frac{A}{A_{eff}}$$

S> 1

Alternately, the shear area of the member can defined as the area of the section which is effective in resisting shear deformation.

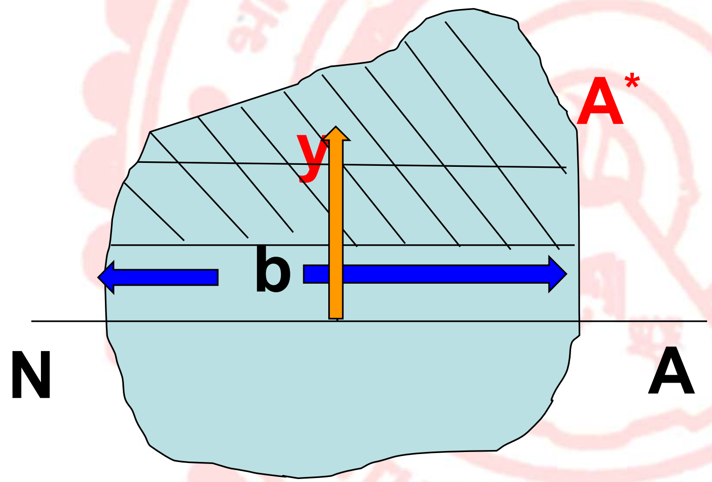
$$U = \iiint_V \frac{1}{2}\sigma\varepsilon dV$$

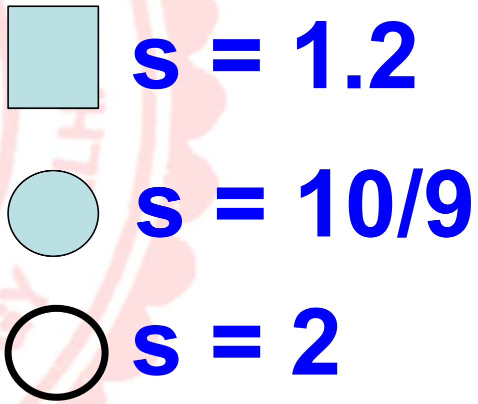
$$U = \int_L \frac{1}{2}\left(\frac{M^2}{EI}\right)dx + \int_L \frac{1}{2}\left(\frac{F^2}{EA}\right)dx$$

$$+ s\int_L \frac{1}{2}\left(\frac{V^2}{GA}\right)dx$$

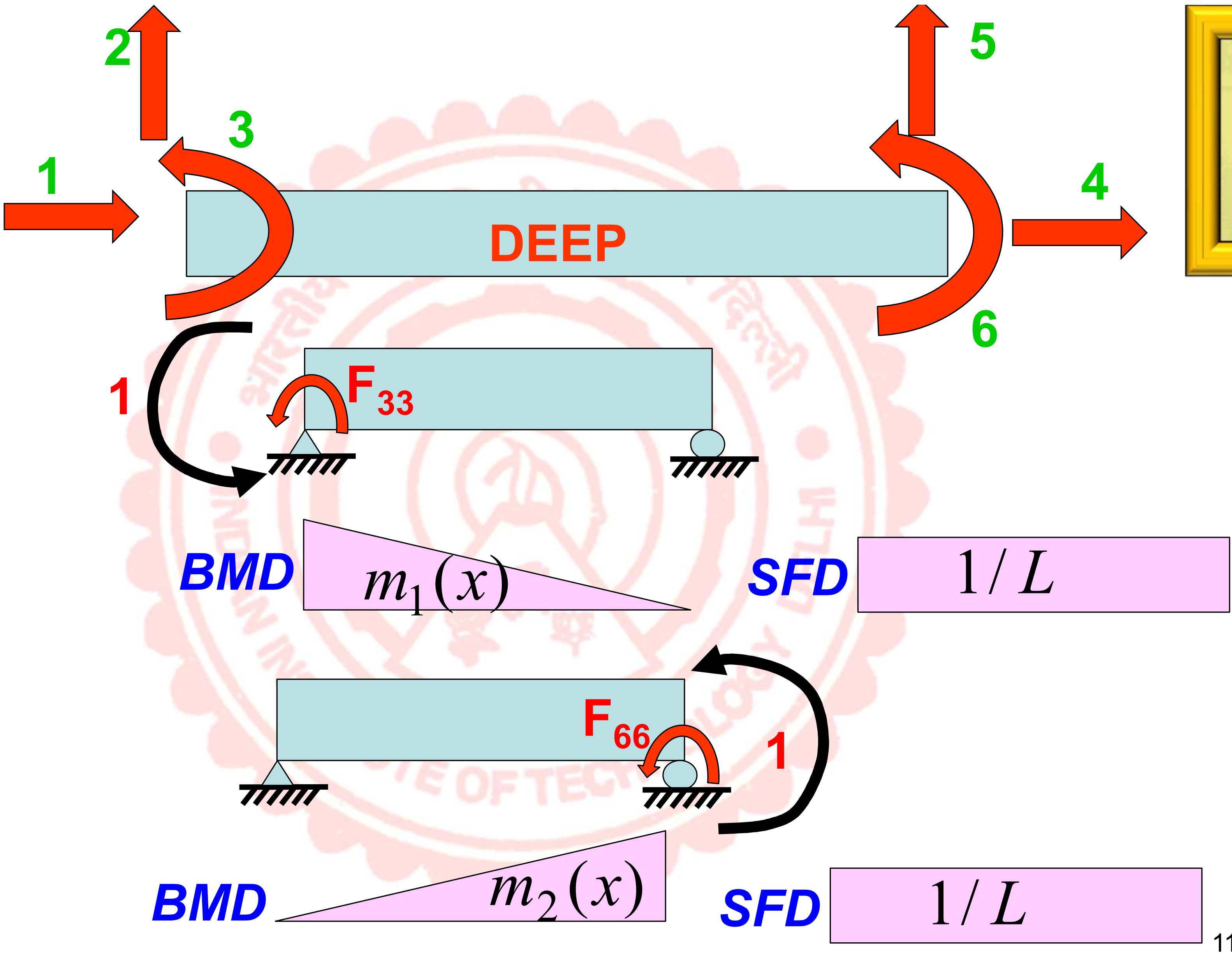
s = Form factor or shear correction factor or shear deformation coefficient

$$s = \frac{A}{I^2}\int_A \left(\frac{Q^2}{b^2}\right) dA$$





Practice: Find ‘s’ for a rectangular section bx2D



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Applying the principle of Virtual Work,

$$F_{33} = \int_L \frac{m_1(x)m_1(x)}{EI}dx + s\int_L \frac{V_1(x)V_1(x)}{GA}dx$$

$$m_1(x) = \left(1 - \frac{x}{L}\right) \qquad V_1(x) = \frac{1}{L}$$

Similarly,

$$F_{63} = \int_L \frac{m_1(x)m_2(x)}{EI}dx + s\int_L \frac{V_1(x)V_2(x)}{GA}dx$$

Similarly get F_{66}

$$F_{66} = \int_L \frac{m_2(x)m_2(x)}{EI}dx + s\int_L \frac{V_2(x)V_2(x)}{GA}dx$$

$$m_2(x) = \frac{x}{L} \qquad V_2(x) = \frac{1}{L}$$

and

$$F_{63} = \int_L \frac{m_1(x)m_2(x)}{EI}dx + s\int_L \frac{V_1(x)V_2(x)}{GA}dx$$

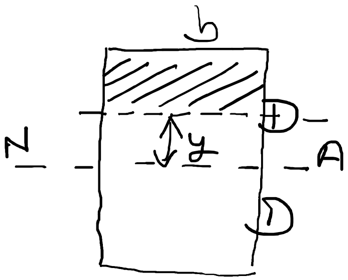
$$F_{66} = \int_L \frac{m_2(x)m_2(x)}{EI}dx + s\int_L \frac{V_2(x)V_2(x)}{GA}dx$$

$$K_{33} = \frac{1}{\left[F_{33} - \dfrac{F_{36}F_{63}}{F_{66}}\right]}$$

From slide18

In computer program, we need not store entire [T], we may simply store [R]

$$[k]_G = [T]^T[k]_L[T] = \begin{bmatrix} R^T & 0 \\ 0 & R^T \end{bmatrix} \underbrace{\begin{bmatrix} K_A & K_B \\ K_B & K_C \end{bmatrix}}_{[K_L]} \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix} = \begin{bmatrix} R^T K_A R & R^T K_B R \\ R^T K_B R & R^T K_C R \end{bmatrix}$$



$$S = \frac{A}{I^2}\int_A \left(\frac{Q^2}{b^2}\right) dA$$

$$A = 2bD$$

$$I = \frac{b(2D)^3}{12} = \frac{2}{3} b D^3$$

$$I^2 = \frac{4}{9} b^2 D^6$$

$$\frac{A}{I^2} = \frac{9}{2\, b D^5}$$

$$Q(y) = b(D-y) \times \left[y + \frac{D-y}{2}\right]$$

$$= \frac{b\left(D^2 - y^2\right)}{2}$$

$$\frac{Q^2(y)}{b^2} = \frac{1}{4}\left(D^2-y^2\right)^2 = \frac{D^4 + y^4 - 2D^2y^2}{4}$$

$$\int_A \frac{Q^2}{b^2}\, \underbrace{dA}_{b\,dy} = \frac{2b}{4}\int_0^D \left(D^4 + y^4 - 2D^2y^2\right) dy = \frac{b}{4}\left[D^4 y + \frac{y^5}{5} - \frac{2D^2y^3}{3}\right]$$

$y = 0$ to $y = D$

$$= \frac{4}{15} b D^5$$

$$S = \frac{4}{15} b D^5 \times \frac{9}{2\, b D^5}$$

$$S = 1.2$$