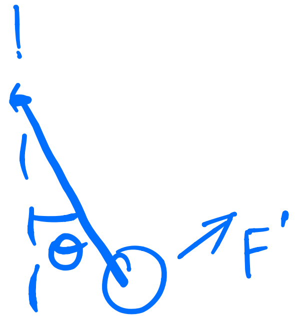
ELL333
LECTURE
03.08.2018

http://web.iitd.ac.in/~shaunak/sen/2018Sem1_ELL333.html ← Lecture pdfs



$$\ddot{\theta} + \frac{g}{l}\sin\theta = \frac{F'}{ml}$$

$x_1 = \theta, \quad x_2 = \dot{\theta}, \quad u = F'/ml$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

general nonlinear form. $\dot{x} = g(x) + Bu$ ↪ $\begin{bmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_n(x) \end{bmatrix}$

$y = h(x,u)$ ← output equation

$$\dot{x} = f(x,u), \quad x \in \mathbb{R}^{n\times 1}, \quad f(x,u) = \begin{bmatrix} f_1(x,u) \\ \vdots \\ f_n(x,u) \end{bmatrix}$$

Why approximate this by its linearization?
One answer: Typically, nonlinear equations cannot be solved in closed form. Sometimes, even when closed form solutions are available, (for example: elliptic integrals for above) they do not provide complete understanding of solution. Linearization provides some partial understanding.

Linearize around an operating point.

For above we were linearizing around $x_1 = 0$ & $x_1 = \pi$

Taylor series provides one easy way to get linearization

$f(x)$, linearize around $x = x_0$

$$\Delta x = x - x_0$$

$$f(x) = f(x_0 + \Delta x) = f(x_0) + \left.\frac{\partial f}{\partial x}\right|_{x=x_0} \cdot \Delta x + \frac{1}{2!} \left.\frac{\partial^2 f}{\partial x^2}\right|_{x=x_0} (\Delta x)^2 + \dots$$

$$\approx f(x_0) + \left.\frac{\partial f}{\partial x}\right|_{x=x_0} \Delta x$$

$\sin x_1$

$\underline{\underline{x_1 = 0}}$ $\qquad \sin x_1 \cong \sin(0) + \left.\cos x_1\right|_{x_1=0} \cdot \Delta x_1$

i.e $x_{1,0} = 0$ $\qquad = \Delta x_1 = x_1$

$\Rightarrow \Delta x_1 = x_1 - 0$

$\underline{x_1 = \pi}$ $\qquad \sin x_1 = \sin(\pi + \Delta x_1)$

i.e $x_{1,0} = \pi$ $\qquad \cong \sin \pi + \left.\cos x_1\right|_{x_1=\pi} \cdot \Delta x_1$

$\Rightarrow \Delta x_1 = x_1 - \pi$ $\qquad = \underline{-\Delta x_1}$

Use these approximations

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = -\frac{g}{l} \sin x_1 + u$$

$$(x_1, x_2) \xrightarrow[\text{transformation}]{\text{co-ordinate}} (x_{1,0} + \Delta x, x_2)$$

$$\frac{d\,\Delta x_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = -\frac{g}{l}(\quad) + u$$

$+\Delta x_1$ for $x_{1,0}=0$

$-\Delta x_1$ for $x_{1,0}=\pi$

WRITE IN THE STATE-SPACE FORM.

$$\frac{d}{dt}\begin{bmatrix}\Delta x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{g}{l} & 0\end{bmatrix}\begin{bmatrix}\Delta x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$$

$$\frac{d}{dt}\begin{bmatrix}\Delta x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ \frac{g}{l} & 0\end{bmatrix}\begin{bmatrix}\Delta x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$$

sign is different

This is general state-space form

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$x_1 = 0, \pi$ are interesting because, they are (for $x_2=0$) equilibrium points (also called fixed points / steady-states)

if system starts at $(x_1, x_2) = (0, 0)$

or $(x_1, x_2) = (\pi, 0)$, then it always stays there (in absence of input) because

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 = x_1(0), \; x_2 = x_2(0).$

For general nonlinear equation, $\dot{x} = f(x)$, equilibrium points are solution of $f(x) = 0$.

Stability . . . .

what are eigenvalues of

$A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix}$, $\quad A = \begin{bmatrix} 0 & 1 \\ g/l & 0 \end{bmatrix}$

for $x_{1,0} = 0$ $\quad$ for $x_{1,0} = \pi$

$\Downarrow$ $\quad$ $\Downarrow$

$\pm j\sqrt{\frac{g}{l}}$ $\quad$ $\pm\sqrt{\frac{g}{l}}$

$\rightarrow$ discussion of stability $\rightarrow$ ~Ch5.