ELL333

LECTURE

04.08.2018

Different notions of stability discussed in last lecture

→ intuitively what is stable / metastable / unstable

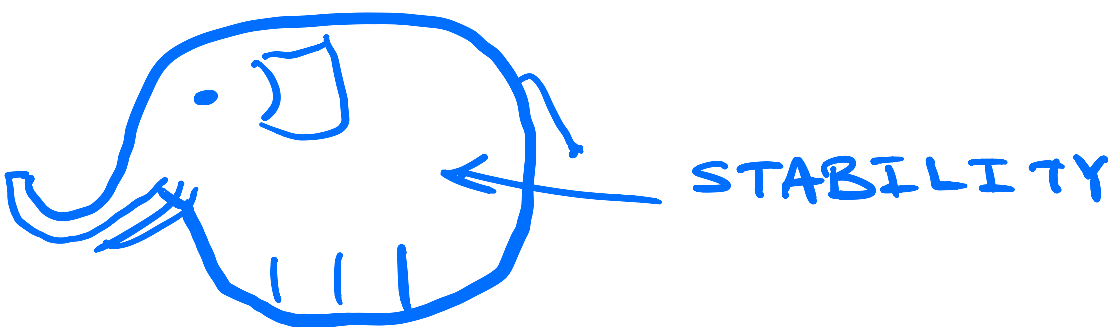
→ loss of rank of a matrix

→ poles are in LHP, once model is available as a transfer function

→ small perturbations from an equilibrium point return to the equilibrium point.

→ restoring force back to equilibrium point

→ potential energy minima



How different aspects are interrelated?

For general 'n' th order system,

$$\dot{x} = f(x, u)$$

$$y = h(x, u)$$

how to linearize?

Two-dimensional example.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}, \quad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\underline{f}(x_1, x_2) = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

One-dimensional example.

$$\dot{x} = f(x)$$

linearize this around $x = x_0$

$$\Delta x = x - x_0$$

$$\frac{d}{dt}\Delta x = \frac{dx}{dt}$$ as x_0 is fixed

$\hookrightarrow \frac{dx_0}{dt} = 0$

For linearizing around trajectory $\frac{dx_0}{dt} \neq 0$

$$= f(x)$$

$$= f(x_0 + \Delta x)$$

$$= f(x_0) + \left.\frac{\partial f}{\partial x}\right|_{x=x_0} \Delta x + \frac{1}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=x_0} (\Delta x)^2 + \cdots$$

for equilibrium point, $f(x_0) = 0$

$$\simeq f(x_0) + \left.\frac{\partial f}{\partial x}\right|_{x=x_0} \Delta x$$

When would $\frac{dx_0}{dt} \neq 0$? make sense ?

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1$$
$\rightarrow$ already linear!

$$\dot{x} = -x^2$$

equilibrium point $x = 0$. Δ $x(0)=0 \Rightarrow x(t)=0$

$$\frac{dx}{dt} = -x^2 \Rightarrow \frac{dx}{x^2} = -dt$$

$$\Rightarrow -\frac{1}{x}\Big|_{x(0)}^{x} = -(t-0)$$

$$\Rightarrow -\frac{1}{x} + \frac{1}{x(0)} = -t$$

$$\Rightarrow \frac{1}{x} = \frac{1}{x(0)} + t$$

$$\Rightarrow x(t) = \frac{1}{\frac{1}{x(0)}+t} = \frac{x(0)}{1+t\,x(0)}$$

linearize ( ) around $x_0(t)$

$$\Delta x = x - x_0(t)$$

$$\frac{d}{dt}\Delta x = \frac{dx}{dt} - \frac{dx_0(t)}{dt}$$

$$= -x^2 - \frac{dx_0(t)}{dt}$$

$$= -(x_0(t) + \Delta x)^2 - \frac{dx_0}{dt}$$

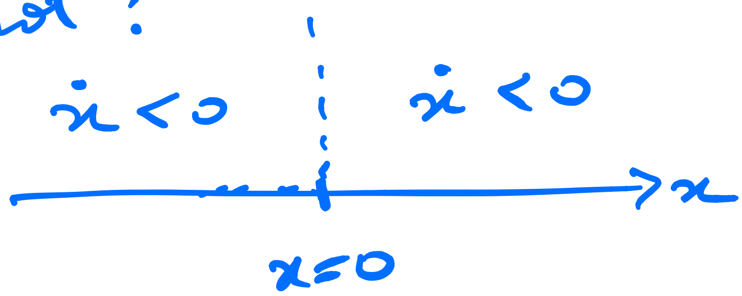
$$= -x_0^2(t) - 2x_0(t)\Delta x - (\Delta x)^2 - \frac{dx_0}{dt}$$

$$\approx -x_0^2(t) - 2x_0(t)\Delta x - \frac{dx_0}{dt}$$

$\rightarrow$ mathematically $\frac{dx_0}{dt}$ need not be zero

$\rightarrow$ other examples?

Related question: Is equilibrium point $x=0$ of $\dot{x} = -x^2$ stable or not?



Two dimensional example - linearization

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}$$

linearize around point $(x_{1,0}, x_{2,0})$

$$\Delta x_1 = x_1 - x_{1,0}$$

$$\Delta x_2 = x_2 - x_{2,0}$$

$$\frac{d}{dt}\Delta x_1 = \frac{dx_1}{dt} - \frac{dx_{1,0}}{dt}$$

$\frac{dx_{1,0}}{dt} = 0$ if $x_{1,0}$ is fixed

$$= f_1(x_1, x_2)$$

$$= f_1(x_{1,0}+\Delta x_1,\ x_{2,0}+\Delta x_2)$$

$$= f_1(x_{1,0}, x_{2,0}) + \left.\frac{\partial f_1}{\partial x_1}\right|_{(x_{1,0},x_{2,0})} \Delta x_1 + \left.\frac{\partial f_1}{\partial x_2}\right|_{(x_{1,0},x_{2,0})} \Delta x_2$$

$$+ \frac{1}{2!}\left\{ \left.\frac{\partial^2 f_1}{\partial x_1^2}\right| (\Delta x_1)^2 + 2\left.\frac{\partial^2 f_1}{\partial x_1 \partial x_2}\right| \Delta x_1 \Delta x_2 + \left.\frac{\partial^2 f_1}{\partial x_2^2}\right| (\Delta x_2)^2 \right\}$$

$$+ \cdots$$

$$\cong f_1(x_{1,0}, x_{2,0}) + \left.\frac{\partial f_1}{\partial x_1}\right| \Delta x_1 + \left.\frac{\partial f_1}{\partial x_2}\right| \Delta x_2$$

Similarly,

$$\frac{d}{dt}\Delta x_2 \simeq f_2(x_{1,0}, x_{2,0}) + \left.\frac{\partial f_2}{\partial x_1}\right| \Delta x_2 + \left.\frac{\partial f_2}{\partial x_2}\right| \Delta x_2$$

$$\Rightarrow \frac{d}{dt}\begin{bmatrix}\Delta x_1 \\ \Delta x_2\end{bmatrix} = \begin{bmatrix} f_1(x_{1,0}, x_{2,0}) \\ f_2(x_{1,0}, x_{2,0})\end{bmatrix} + \underbrace{\left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\right|_{(x_{1,0},x_{2,0})}}_{\text{Jacobian}} \begin{bmatrix}\Delta x_1 \\ \Delta x_2\end{bmatrix}$$

$= \begin{bmatrix}0 \\ 0\end{bmatrix}$ if $(x_{1,0}, x_{2,0})$ is equilibrium point

↳ check this?