ELL333

MULTIVARIABLE CONTROL

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Example of non-uniqueness:

$$\frac{dx}{dt} = \sqrt{x} \qquad x(0) \text{ - initial condition}$$

$\Downarrow$ integrating

$$\sqrt{x(t)} = \frac{t}{2} + \sqrt{x(0)}$$

Construction from textbook.

for $x(0)=0$, consider

$$x(t) = \begin{cases} 0, & 0 \le t < a \\ \left(\frac{t}{2} - a\right)^2, & t \geq a \end{cases}$$

Does it solve the above equation?

$0 \le t < a$, $\quad \frac{dx}{dt} = 0 = \sqrt{x}$ ✓

$t \geq a$, $\quad \frac{dx}{dt} = 2\left(\frac{t}{2} - a\right) \times \frac{1}{2} = \left(\frac{t}{2} - a\right) = \sqrt{x}$ ✓

Point here: For this initial condition, it is possible to create a family of solutions (for any $a > 0$) that satisfy the equation $\Rightarrow$ non-uniqueness

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How about constructing similar solution for $\frac{dx}{dt} = -x$? $x(0) = 0$

$$\rightarrow x(t) = \begin{cases} 0, & 0 \le t < a \\ c e^{-t}, & t \geq a \end{cases}$$

This has to be zero.

Existence and uniqueness properties depend on the continuity properties of the RHS of $\dot{x} = f(x)$. i.e the function f.
In particular, 'LIPSCHITZ' continuity is a condition on '$f(x)$' to get existence and uniqueness of solutions.

For linear (or linearized) state-space model

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du,$$

$x(0)$ – initial condition

solution exists and is unique.

So, how to solve these equations

$$\boxed{\dot{x} = Ax} + Bu$$
$$y = Cx + Du$$

First, we will solve this part, assuming no input, $u = 0$

natural response
(or) initial condition response
(or) homogeneous solution

Second, we will solve the entire part, including the effect of input i.e particular solution
(or) forced response

General way to solve $\dot{x} = Ax$ is to generalize what is done when A is a scalar to the case when A is a matrix

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Is there some other way?

$$\dot{x} = a\,x, \quad a \in \mathbb{R}, \quad x(0)$$

$$x(t) = \underbrace{x(0)}_{\text{initial condition}}\, \underbrace{e^{at}}_{\text{exponential}}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{bmatrix}$$

Common solution is to generalize exponential to a <u>matrix exponential</u>

$$e^{x} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots$$

Define matrix exponential; $\rightarrow$ for square matrices X

$$e^{X} \triangleq I + X + \frac{1}{2!}X^2 + \cdots + \frac{1}{k!}X^k + \cdots$$

$I \rightarrow$ Identity $\qquad X^2 = X.X, \quad X^k = X^{k-1}.X$

for scalars, $x = at$

for matrices, $X = At$

$$e^{At} = I + tA + \frac{t^2}{2!}A^2 + \cdots + \frac{t^k}{k!}A^k + \cdots$$

Matrix Exponential e^{At} + Initial condition vector $x(0)$ + System equation $\dot{x} = Ax$ = $\boxed{\text{Solution?}}$

Rough: Will series converge if eigenvalues of A are negative?

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Questions: about convergence of A.