ELL333

MULTIVARIABLE CONTROL

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Scalar Case

$\dot{x} = a x$, $x(0)$ $\Rightarrow$ $x(t) = x(0) e^{at}$

General case

$\dot{x} = A x$, $x(0)$ $\Rightarrow$ $x(t) = ?$

matrix

vector

defined matrix exponential

$$\underline{\underline{e^{At}}} = I + At + A^2 \frac{t^2}{2!} + \dots + A^k \frac{t^k}{k!} + \dots$$

$\dot{x} = A x$, $x(0)$, e^{At}

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{bmatrix}_{n \times 1} \qquad \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}_{n \times n}$$

How to combine to get solution?

(I) $x(t) = \alpha e^{At}$ ← careful about dimensions

$n \times 1$, $x(0)$

(II) $\underbrace{x(0)^T}_{1 \times n} e^{At}$

(III) $e^{At} x(0)$ → solution?

Claim: Solution of $\dot{x} = Ax$, $x(0)$ is

$$x(t) = e^{At} x(0)$$

How to check if this is solution?

at $t=0$, $x(t) = e^{At} x(0) = \underbrace{e^{A.0}}_{I + A.0 + A.\frac{0^2}{2!} \dots} x(0) = x(0)$

does it satisfy $\dot{x} = Ax$

$$\dot{x} = \frac{d}{dt} x = \left[\frac{d}{dt} e^{At}\right] x(0)$$

$$e^{At} = I + tA + \frac{t^2}{2!} A^2 + \dots + \frac{t^k}{k!} A^k + \dots$$

$$\frac{d}{dt} e^{At} = 0 + A + tA^2 + \dots + \frac{t^{k-1}}{(k-1)!} . A^k + \dots$$

$$= A \left[I + tA + \dots + \frac{t^{k-1}}{(k-1)!} A^{k-1} + \dots \right]$$

$$= A e^{At}$$

$$\dot{x} = A \left[e^{At} x(0)\right] \rightarrow x(t)$$

This is the 'existence' part. Solution exists!

What about 'uniqueness'?

<u>Try</u> Suppose two solutions x_1, x_2, $x_1 \neq x_2$

$x_1(t) = e^{At} x(0)$, $= e^{At} x_1(0)$ $\leftarrow$ $x_1(0) = x(0)$, $\dot{x}_1 = Ax_1$ and $x_2(0) = x(0)$ and $\dot{x}_2 = Ax_2$ $\rightarrow$ $x_2(t) = e^{At} x_2(0) = e^{At} x(0)$

define $y = x_1 - x_2$, $y(0) = 0$

$$\dot{y} = \dot{x}_1 - \dot{x}_2 = Ax_1 - Ax_2 = A\overbrace{(x_1 - x_2)}^{y}$$

$\Rightarrow \dot{y} = Ay$ $\Rightarrow$ $y(t) = e^{At} . y(0) = 0$

$\Rightarrow$ $x_1(t) = x_2(t) \Rightarrow$ solution is unique.

$\dot{y} = Ay$, $y(0) = 0$.

What is the equilibrium point of the system $\dot{y} = Ay$?

$\Rightarrow$ $y = 0$ or any $y \in \text{nullspace}(A)$

$\hookrightarrow$ one way of showing $y(t) = 0$

$\Rightarrow x_1(t) = x_2(t)$

Solution of $\dot{x} = Ax$, $x(0)$ is
$$x(t) = e^{At} x(0).$$

How to compute e^{At}?

For what matrices is e^{At} easy to compute?

Diagonal matrices

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

What is $e^{(\cdot)t} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$

$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad A^2 = \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix}, \quad \ldots \quad A^k = \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix}$$

$$e^{At} = I + At + A^2 \frac{t^2}{2!} + \ldots A^k \frac{t^k}{k!} + \ldots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 t & 0 \\ 0 & \lambda_2 t \end{bmatrix} + \begin{bmatrix} \lambda_1^2 t^2/2! & 0 \\ 0 & \lambda_2^2 t^2/2! \end{bmatrix} + \ldots \begin{bmatrix} \lambda_1^k t^k/k! & 0 \\ 0 & \lambda_2^k t^k/k! \end{bmatrix} + \ldots$$

$$= \begin{bmatrix} 1 + \lambda_1 t + \frac{\lambda_1^2 t^2}{2!} + \dots \frac{\lambda_1^k t^k}{k!} + \dots & 0 \\ 0 & 1 + \lambda_2 t + \frac{\lambda_2^2 t^2}{2!} + \frac{\lambda_2^k t^k}{k!} + \dots \end{bmatrix}$$

$$= \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$$

$$e^{At} = ?$$

$$A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \rightsquigarrow \text{Jordan Canonical Form.}$$

$$e^{At} = ?$$