ELL 333

MULTIVARIABLE CONTROL

11.09.2018

DYNAMICS > MATRIX EXPONENTIAL

Example: $\ddot{x} = u$, $x(0), \dot{x}(0)$

'Double Integrator'

For $u=0$, $\ddot{x} = 0$, $x(0), \dot{x}(0)$

So what is the solution?

$\frac{d}{dt}\dot{x} = 0 \overset{\text{INTEGRATE}}{\Rightarrow} \dot{x}(t) =$ a constant, whose value is $\dot{x}(0)$ by initial condition

Now, $\frac{dx}{dt} = \dot{x}(0)$

INTEGRATE $\Rightarrow$ $x(t) = \dot{x}(0)t +$ another constant, whose value is $x(0)$

$\Rightarrow$ $x(t) = \dot{x}(0)t + x(0)$ ← solution from direct integration

This is one way to find solution.

Another way is to convert into a state-space representation and use matrix exponential.

STATE SPACE of $\ddot{x} = u$ ⟶ $x_1 = x$, $x_2 = \dot{x}$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Solution of homogeneous part using matrix exponential is $e^{At}\begin{bmatrix} x(0) \\ \dot{x}(0) \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

What is e^{At}?

$$e^{At} = I + At + A^2\frac{t^2}{2!} + \dots + A^k\frac{t^k}{k!} + \dots$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$e^{At} = I + At = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

Solution: $e^{At}\begin{bmatrix} x(0) \\ \dot{x}(0) \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x(0) \\ \dot{x}(0) \end{bmatrix}$

$$= \begin{bmatrix} x(0) + \dot{x}(0) \cdot t \\ \dot{x}(0) \end{bmatrix}$$

$\Rightarrow$ $\underline{x_1(t) = x(0) + \dot{x}(0)t}$

$\hookrightarrow$ matches the solution obtained from direct integration.

Consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

This has two eigenvalues, both are '0'

What are the corresponding eigenvectors?

$\hookrightarrow$ eigenvector is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\qquad A - \lambda I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Recall definition: $v (\neq 0)$ is an eigenvector of $\underline{n \times n}$ matrix A with eigenvalue λ if $Av = \lambda v$

Consider $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

eigenvalues: $0, \quad 1$

eigenvectors: $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Two eigenvectors, with repeated eigenvalues

What about $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

eigenvalues: '1', '1'

eigenvectors: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$Av = \lambda v$

$(A - \lambda I)v = 0$

$A - \lambda I$

$\lambda = 1, A = I$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Going back to red example,

$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

eigenvalues: 1, 0

eigenvectors: $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

How to diagonalize this?

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \xrightarrow[\text{'T'} \; 2\times 2]{\text{Co-ordinate transformation}} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ diagonal}$$

T is related to eigenvectors

$T = ?$