ELL333

MULTIVARIABLE CONTROL

12.09.2018

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How to compute matrix exponential to solve $\dot{x} = Ax$?

$$\begin{tikzcd} A \arrow[r, "\text{co-ordinate transformation}"] \arrow[d, "\text{what we want}"', "\text{possibly hard}"] & \text{diagonal (or diagonal-like)} \arrow[d, "\text{(easy)}"] \\ \text{matrix exponential of } A & \text{matrix exponential of diagonal} \arrow[l, "\text{invert the}"', "\text{co-ordinate transformation}"] \end{tikzcd}$$

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Computed eigenvalues, eigenvectors yesterday.
how does this help to get the co-ordinate transformation

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Suppose we have '2' distinct eigenvalues for a $2\times 2$ matrix $A$,

$$Av_1 = \lambda_1 v_1 \qquad , \qquad v_1, v_2 (\neq 0)$$
$$Av_2 = \lambda_2 v_2 \qquad \qquad \lambda_1 \neq \lambda_2$$

$$\begin{bmatrix} Av_1 & Av_2 \end{bmatrix}_{2\times 2} = \begin{bmatrix} \lambda_1 v_1 & \lambda_2 v_2 \end{bmatrix}_{2\times 2}$$

Diagonal

$$A_{2\times 2} \underbrace{\begin{bmatrix} v_1 & v_2 \end{bmatrix}_{2\times 2}}_{T} = \underbrace{\begin{bmatrix} v_1 & v_2 \end{bmatrix}}_{2\times 2} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$AT = T\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}$$

$$T^{-1} \times [\ \Rightarrow T^{-1}AT = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}$$

or $A = T\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}T^{-1}$

$e^{At}$ : how does this help?

So, what is relation between $e^{At}$ and $e^{Dt}$, $D = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}$ [ $e^{Dt}$ is easy $= \begin{bmatrix}e^{\lambda_1 t} & 0\\ 0 & e^{\lambda_2 t}\end{bmatrix}$

Claim $e^{At} = T e^{Dt} T^{-1}$

if $A = TDT^{-1}$, $T^{-1}AT = D$

$$e^{At} = I + A + A^2\frac{t^2}{2!} + \dots + A^k\frac{t^k}{k!} + \dots$$

$$= I + TDT^{-1} + (TD\underbrace{T^{-1})(T}_{=I}DT^{-1})\frac{t^2}{2!} + \dots$$

$$+ (TD\underbrace{T^{-1})(T}_{I}D\underbrace{T^{-1})\dots}(TDT^{-1})\frac{t^k}{k!} + \dots$$

$$= \underset{T\uparrow\ \ T^{-1}\uparrow}{I} + TDT^{-1} + TD^2T^{-1}\frac{t^2}{2!} + \dots + TD^kT^{-1}\frac{t^k}{k!} + \dots$$

$$= T\underbrace{\left[I + D + D^2\frac{t^2}{2!} + \dots + D^k\frac{t^k}{k!} + \dots\right]}_{e^{Dt}}T^{-1}$$

Above should work as long as there are two linearly independent eigenvectors i.e even if $\lambda_1 = \lambda_2$.

From last lecture?

Going back to red example,

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

eigenvalues : 1, 0

eigenvectors : $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

How to diagonalize this?

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \xrightarrow[\text{'T'} \; 2\times 2]{\text{Co-ordinate transformation}} \overset{\text{diagonal}}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} = D$$

T is related to eigenvectors

$T = ?$ $\quad e^{Dt}$ ? $\quad e^{At}$ ?

$\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$, $T^{-1}$, $\begin{bmatrix} e^t & 0 \\ 0 & 1 \end{bmatrix}$, $T e^{Dt} T^{-1}$

$\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ X

$\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$

This covers a lot of matrices for which matrix exponential needed and have distinct eigenvalues (or n linearly independent eigenvectors)

What about this other case?

: Consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

This has two eigenvalues, both are '0'

What are the corresponding eigenvectors?

↳ eigenvector is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad A - \lambda I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

This is a degenerate case and related to the Jordan Canonical Form.

Workaround is to define a 'generalized eigenvector' And use it to complete a basis

$$(A - \lambda I)^2 v = 0$$