EE333
Multivariable Control
14.09.2018

$$J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$ ← an example of a 2×2 Jordan Canonical Form. for $\lambda=0$, this is the double integrator system

$$e^{tJ} = ?$$

Compute using series.

Reference: Nineteen Dubious Ways to Compute the Exponential of a Matrix, Twenty-Five Years Later*
SIAM Review, Vol. 45, No. 1
Cleve Moler, Charles van Loan

$$e^{tJ} = I + tJ + \frac{t^2}{2!}J^2 + \cdots + \frac{t^k}{k!}J^k + \cdots$$

$$J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

$$J^2 = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{bmatrix}$$

$$J^3 = \begin{bmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{bmatrix}\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} \lambda^3 & 3\lambda^2 \\ 0 & \lambda^3 \end{bmatrix}$$

$$J^4 = \begin{bmatrix} \lambda^3 & 3\lambda^2 \\ 0 & \lambda^3 \end{bmatrix}\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} \lambda^4 & 4\lambda^3 \\ 0 & \lambda^4 \end{bmatrix}$$

In general, $J^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} \\ 0 & \lambda^k \end{bmatrix}$

$$e^{tJ} = \begin{bmatrix} 1+t\lambda+\frac{t^2}{2!}\lambda^2+\dots+\frac{t^k}{k!}\lambda^k+\dots & 0+t\cdot 1+\frac{t^2}{2!}\cdot 2\lambda+\frac{t^3}{3!}\cdot 3\lambda^2+\dots\frac{t^k}{k!}k\lambda^{k-1}\dots \\ 0 & \text{same} \end{bmatrix}$$

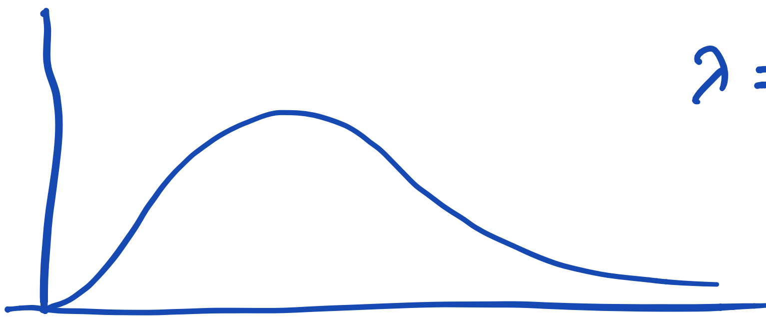
$$0+t\cdot 1+\frac{t^2}{2!}\cdot 2\lambda+\dots+\frac{t^k}{k!}k\lambda^{k-1}+\dots$$

$$= \frac{d}{d\lambda}e^{\lambda t} = \boxed{t e^{\lambda t}}$$

$$e^{\lambda t} = 1+t+\frac{t^2}{2!}\lambda^2+\dots+\frac{t^k}{k!}\lambda^k+\dots$$

<u>$t e^{\lambda t}$</u>

$\lambda = -1$



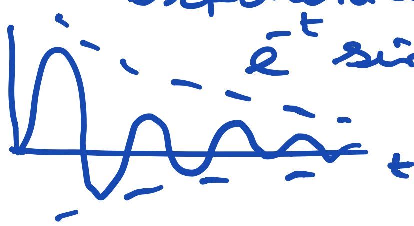
t

pure exponential decay

e^{-t}

sinusoidal + exponential decay

$e^{-t}\sin\omega t$



SUMMARY

We want to compute matrix exponential e^{At}, to get $e^{At}x(0)$ which is solution of $\dot{x} = Ax$, $x(0)$.

One way : Matrix Decomposition

A

if there are eigenvectors which form a basis *

primary decomposition theorem

invertible

$$A = T^{-1} D T$$

diagonal

$$\Rightarrow e^{At} = T^{-1} e^{Dt} T$$

this is easy

* a sufficient condition is that A has distinct eigenvalues

if the eigenvectors don't form a basis

$$A = T^{-1} J T$$

Jordan Canonical Form

$$\Rightarrow e^{At} = T^{-1} e^{Jt} T$$

this is also easy.

$$J_{2\times 2} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

$$J_{3\times 3} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

for example

$$J = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3 \end{bmatrix}$$

6×6

How to do this conversion?

generalized eigenvectors, secondary decomposition theorem.