ELL333

MULTIVARIABLE CONTROL

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$$\dot{x} = Ax + Bu$$
$$y = Cx + Du \quad , x(0)$$

We wanted to solve dynamics of this one

Solution

$$x(t) = e^{At} x(0)$$

compute matrix exponential

Thm: Any $n \times n$ matrix 'A' can be written as

$$A = TJT^{-1}$$

J → Jordan Canonical Form (Diagonal matrix is a kind of Jordan Form)

e^{Jt} is easy to calculate

$$\Rightarrow \quad e^{At} = T e^{Jt} T^{-1}$$

This is just solution of $\dot{x} = Ax$.

What about $\dot{x} = Ax + Bu$

$$\dot{x} = ax + bu \quad \leftarrow \text{scalar case}$$

$$x(t) = ?$$

Solution: $\quad \dot{x} - ax = bu$

Then, multiply by e^{-at} (method of Integrating Factors

$$e^{-at}\dot{x} - a e^{-at} x = b u e^{-at}$$

$$\frac{d}{dt} x(t) e^{-at} = b u(t) e^{-at}$$

$$\Rightarrow \int_0^t d\, x(t') e^{-at'} = \int_0^t b\, u(t') e^{-at'} dt'$$

$$\Rightarrow x(t) e^{-at} - x(0) = \int_0^t b\, u(t') e^{-at'} dt'$$

$$\Rightarrow x(t) e^{-at} = x(0) + \int_0^t b\, u(t') e^{-at'} dt'$$

multiply by e^{at}

$$\Rightarrow x(t) = x(0) e^{at} + \int_0^t b\, u(t') e^{a(t-t')} dt'$$

$$x(t) = e^{At} x(0) \qquad ?$$

Let's "generalize" the scalar solution and then check it.

$$\int_0^t e^{A(t-t')} B\, u(t') dt'$$

$B_{n\times p}$, $u_{p\times 1}$, $e^{A(t-t')}_{n\times n}$

Candidate solution for $\dot{x} = Ax + Bu$.

$$x(t) = e^{At} x(0) + \int_0^t e^{A(t-t')} B\, u(t') dt'$$

Check that this solution satisfies $\dot{x} = Ax + Bu$.

$$\frac{dx}{dt} = \underbrace{\frac{d}{dt} e^{At} x(0)}_{A e^{At} x(0)} + \frac{d}{dt} \int_0^t e^{A(t-t')} B\, u(t') dt'$$

$A e^{At} x(0)$ [using $\frac{d}{dt} e^{At} = A e^{At}$]

$$\frac{d}{dt}\int_0^t e^{A(t-t')} Bu(t')\,dt'$$

(use Leibniz Formula)

$$= e^{A(t-t')}Bu(t')\Big|_{t'=t}\frac{d}{dt}.t$$

$$- e^{A(t-t')}Bu(t')\Big|_{t'=0}\frac{d}{dt}.0$$

$$+ \int_0^t \underbrace{\left[\frac{\partial}{\partial t}e^{A(t-t')}Bu(t')\right]}\,dt'$$

$$= \left[\frac{\partial}{\partial t}e^{At}\right]e^{-At'}Bu(t')$$

$$= A\,e^{At}.\,e^{-At'}Bu(t')$$

$$= \underbrace{e^{A0}}_{I}Bu(t) - 0 + A\int_0^t e^{A(t-t')}Bu(t')\,dt'$$

Overall,

$$\frac{dx}{dt} = A\underbrace{\left(e^{At}x(0) + \int_0^t e^{A(t-t')}Bu(t')\,dt'\right)}_{x(t)} + Bu(t)$$

$$\frac{dx}{dt} = Ax + Bu\text{, this satisfies the equation.}$$

$\therefore$ solution of $\dot{x} = Ax + Bu$, $y = Cx + Du$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-t')}Bu(t')\,dt'$$

$$y(t) = Cx(t) + Du(t)$$

For solutions of first-order, second-order systems, typically we plot them for different system parameters (time constants; damping co-efficient, natural frequency).

For higher order systems, there may be more possible responses. Still, solution above is a starting point to study important system properties such as

1. Stability (A)
2. Controllability (A, B)
3. Observability (C, A)

as well as other aspects of solution.