ELL333

MULTIVARIABLE CONTROL

25.09.2018

Definition of stability $\rightarrow$ Stability in the sense of LYAPUNOV.

$\dot{x} = f(x).\ x(0) = a$

Denote the solution as $x(t;a)$

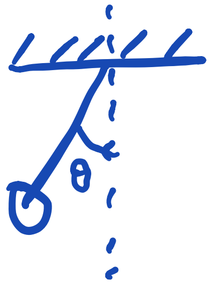
[$x(t;a)$ could be an equilibrium point or any other solution.]

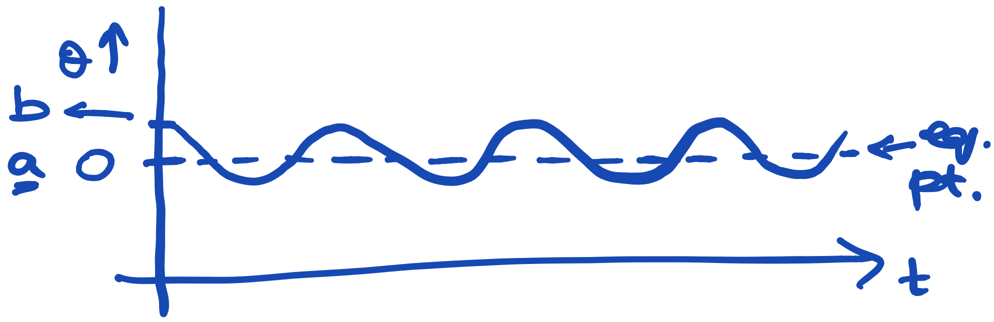
$x(t;a)$ is said to be stable if

$\forall\, \epsilon > 0\ \exists\, \delta > 0$ such that

$$|b-a| < \delta \Rightarrow |x(t;b) - x(t;a)| < \epsilon \ \forall t > 0.$$

example!



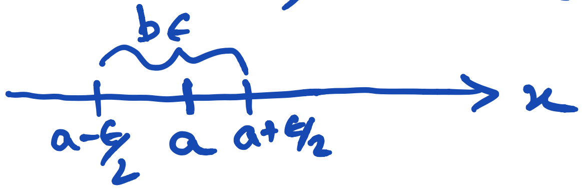


no friction

This is stable in the sense of Lyapunov.

possibly simpler example.

$\dot{x} = 0 \quad \Rightarrow \quad x = x(0).$



Suppose $\epsilon = 1$

Choose a $\delta = \epsilon/2$

$$|b-a| < \epsilon/2 \Rightarrow |\underbrace{x(t;b)}_{\hookrightarrow b} - \underbrace{x(t;a)}_{\hookrightarrow a}| < \frac{\epsilon}{2} < \epsilon \ \forall t > 0$$

Trying to apply the definition here $\rightarrow$ equations of motion. ?

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$ [See lecture of 1 Aug

$x_1 = \theta$, $x_2 = \dot{\theta}$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l}\sin x_1$$
[state space representation

equilibrium points are

$\dot{x}_1 = 0 \Rightarrow x_2 = 0 = \dot{\theta}$

$\dot{x}_2 = 0 \Rightarrow \sin x_1 = 0 \Rightarrow x_1 = 0$ or π (θ)

Look at $\theta = 0$, $\dot{\theta} = 0$, and linearization

$x_1 = 0$, $x_2 = 0$

linearization is $\dot{x} = Ax$, where

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix}$$

[using the definition of Jacobian matrix

basically $\sin\theta \approx \theta$

Solution

$\rightarrow$ matrix exponential

$\hookrightarrow$ solve directly $\rightarrow$ $\boxed{\ddot{\theta} + \frac{g}{l}\theta = 0}$

$$\theta(t) = A\sin\omega t + B\cos\omega t, \quad \omega = \sqrt{\frac{g}{l}}$$

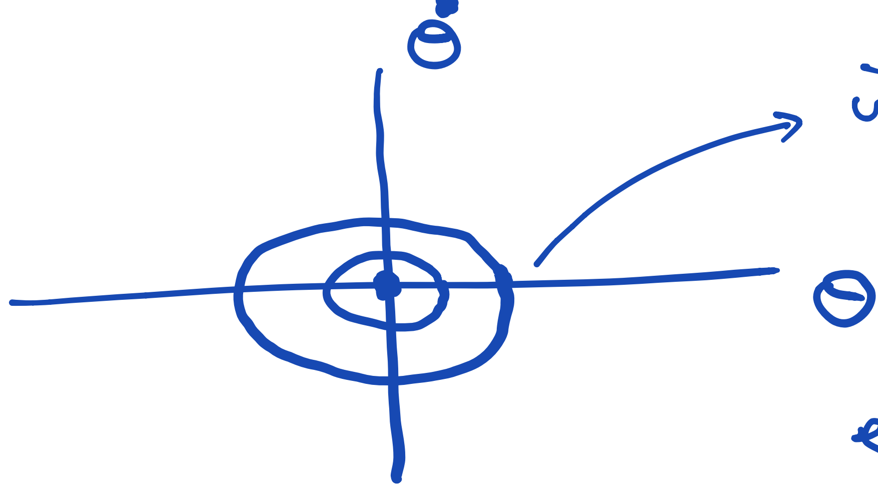
$$\dot{\theta}(t) = A\omega\cos\omega t + B(-\omega)\sin\omega t$$

Put $t = 0 \Rightarrow \theta(0) = B$

and $\dot{\theta}(0) = A\omega$

$\therefore$ Solution $\theta(t) = \theta(0)\cos\omega t + \frac{\dot{\theta}(0)}{\omega}\sin\omega t$

$$\dot{\theta}(t) = \dot{\theta}(0)\cos\omega t - \omega\,\theta(0)\sin\omega t$$



initial condition

$\theta(0) \neq 0$

$\dot{\theta}(0) = 0$

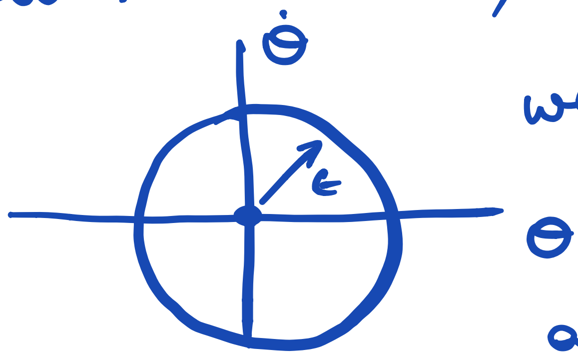
$\Rightarrow \theta(t) = \theta(0)\cos\omega t$

$\dot{\theta}(t) = -\omega\,\theta(0)\sin\omega t$

equation of ellipse

$$\left(\frac{\theta(t)}{\theta(0)}\right)^2 + \left(\frac{\dot{\theta}(t)}{-\omega\,\theta(0)}\right)^2 = 1$$

Given $\epsilon > 0$, $x(t;a) = (0,0)$



we want $|x(t;b) - \underbrace{x(t;a)}_{(0,0)}| < \epsilon$

and we want to find δ such that $|b-a| < \delta$

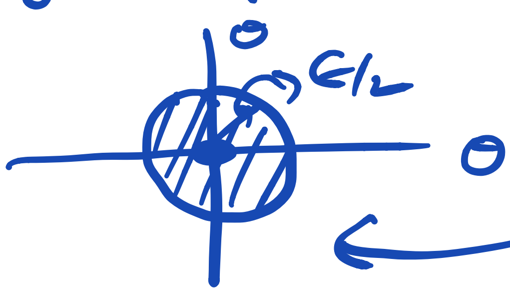
$\Rightarrow |x(t;b) - x(t;a)| < \epsilon.$

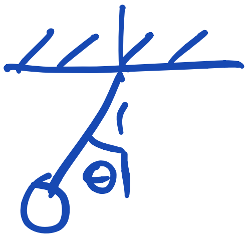
How do we find δ?

Suppose $\omega = 1$.

Could we choose $\delta = \frac{\epsilon}{2}$?

any b such that $|b-a| < \epsilon/2$

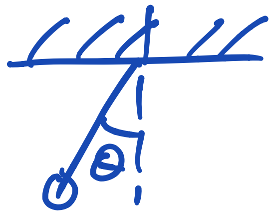




no friction

$\Downarrow$

Lyapunov stable



with friction

asymptotically stable

= Lyapunov stable

$+ \quad \lim_{t\to\infty} |x(t;b) - x(t;a)| = 0$