ELL333

MULTIVARIABLE CONTROL

23.10.2018

Relation between inertia and mass, radius... in bicycle?

Example: Changing eigenvalues using feedback $u=-Kx$?

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x \quad \longrightarrow \text{unstable} \quad , \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

add control input

$$\dot{x} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{2\times1}}_{B} u_{1\times1}$$

Can this be made stable?

In particular, using a linear feedback,

$$u = -K_{1\times2}\, x_{2\times1} = -[k_1 \;\; k_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -k_1x_1 - k_2x_2$$

can this be made stable by choosing K?

$\Rightarrow$ Have to check eigenvalues of $A-BK$:

roots: $1, \; 1-k_1-k_2$

$$A-BK = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}[k_1 \;\; k_2]$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} k_1 & k_2 \\ k_1 & k_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-k_1 & -k_2 \\ -k_1 & 1-k_2 \end{bmatrix}$$

characteristic polynomial $\}$: $(\lambda-1+k_1)(\lambda-1+k_2) - k_1k_2 = 0$

$$\Rightarrow (\lambda-1)^2 + k_1(\lambda-1) + k_2(\lambda-1) + \cancel{k_1k_2} - \cancel{k_1k_2} = 0$$

$$\Rightarrow (\lambda-1)(\lambda-1+k_1+k_2) = 0$$

Not always feedback $u=-Kx$ can change eigenvalues of the 'A' matrix to desired locations

---

Question: When can $u=-Kx$ change eigenvalues of the system and when not?

Consider a general $n=2$ system with one input

$$\dot{x} = \underset{2\times 2}{A}x + \underset{2\times 1}{B}u, \quad u = -\underset{1\times 2}{K}x$$

$$\Rightarrow \quad \dot{x} = \underbrace{(A-BK)}\,x$$

Can eigenvalues of this be placed anywhere through choice of K?

Desired eigenvalues : μ_1, μ

$\Rightarrow$ Characteristic polynomial : $(\lambda-\mu_1)(\lambda-\mu_2) = 0$

$$\lambda^2 - (\mu_1+\mu_2)\lambda + \mu_1\mu_2 = 0$$

redefine : $\lambda^2 + a_1\lambda + a_2 = 0$

[Cayley-Hamilton Theorem: A matrix satisfies its own characteristic equation]

$$\Rightarrow \quad (A-BK)^2 + a_1(A-BK) + a_2 I = 0$$

$$(A-BK)^2 = (A-BK)(A-BK)$$
$$= A^2 - ABK - BKA + BKBK$$

$$\Rightarrow \quad A^2 - ABK - BKA + BKBK$$
$$+ a_1 A - a_1 BK + a_2 I = 0$$

$$\Rightarrow \quad A^2 + a_1 A + a_2 I = ABK + BKA - BKBK + a_1 BK$$

$$\Rightarrow \quad ABK + BKA - BKBK + a_1 BK = A^2 + a_1 A + a_2 I$$

$$\Rightarrow \quad \underbrace{[B \;\vdots\; AB]_{2\times 2}}_{\text{known}} \underbrace{\begin{bmatrix} KA - KBK + a_1 K \\ K \end{bmatrix}_{2\times 2}}_{K \text{ unknown}} = \underbrace{A^2 + a_1 A + a_2 I}_{\text{known}}.$$

If $[B \;\; AB]$ is invertible
(same as $\text{rank}\{[B \;\; AB]\} = 2$)

$$\Rightarrow \quad \begin{bmatrix} KA - KBK + a_1 K \\ K \end{bmatrix} = [B \;\; AB]^{-1} (A^2 + a_1 A + a_2 I)$$

$$\Rightarrow \quad K = [0 \;\; 1] \; [B \;\; AB]^{-1} \; (A^2 + a_1 A + a_2 I)$$

Two things
1. Does K exist? Yes, if $\text{rank}\{[B \;\; AB]\} = 2$
2. What K?

Check $\text{rank}\{[B \;\; AB]\}$ in earlier example?

For general n^{th} order system, with single input

$$K = [0 \; 0 \ldots 1]_{1\times n} \; [B \;\; AB \;\ldots\; A^{n-1}B]^{-1} \; \phi(A),$$

where $\phi(\lambda) = 0$ is the polynomial whose roots are the desired eigenvalues.

- This is the Ackermann's Formula.
- A solution K exists if $\text{rank}\{\underbrace{[B \;\; AB \cdots A^{n-1}B]}_{\text{Controllability matrix}}\} = n$

How does this change for a multiple input case?