ELL333

Multivariable Control

24.10.2018

Controllability

Canonical Form for Eigenvalue Assignment

$$A = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}_{3\times 3}$$

$n \times n$

$$\begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

char poly(A) : $\det(\lambda I - A) = 0$

$\Rightarrow \quad \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$

A is companion matrix to this ( )

Charecter-istic polynomial

$\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n = 0$

Canonical form of state-space model

$\dot{x} = Ax + Bu$, $\quad B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1}$ (or) $B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_{3\times 1}$

single input

Control is

$u = -Kx$, $\quad K = [k_1 \ k_2 \ k_3]_{3\times 1}$

What is $A - BK$?

$$= \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}_{3\times 3} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} [k_1 \ k_2 \ k_3]$$

$$= \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} k_1 & k_2 & k_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -(a_1+k_1) & -(a_2+k_2) & -(a_3+k_3) \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Char poly $(A-BK)$: $\lambda^3 + \underbrace{(a_1+k_1)}\lambda^2 + \underbrace{(a_2+k_2)}\lambda + \underbrace{(a_3+k_3)} = 0$

Known,

if desired eigenvalues are specified.

If a system can be transformed into this canonical form, then the controller gains can be assigned by examining the matrix entries.

How to transform a system into this form? (When is it possible?)

$$\dot{x} = Ax + Bu$$

We are looking at linear transformations of the form $x_c = Tx$, T is invertible matrix

$(x = T^{-1}x_c)$

$$\begin{aligned} \dot{x}_c &= T\dot{x} \\ &= T(Ax + Bu) \\ &= TAx + TBu \\ &= TAT^{-1}x_c + TBu \end{aligned}$$

$$\Rightarrow \dot{x}_c = \underbrace{(TAT^{-1})}_{A_c} x_c + \underbrace{(TB)}_{B_c} u$$

When can a system be transformed into canonical form, $A_c = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \end{bmatrix}$, $B_c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$$W = \left[B \,\vdots\, AB \,\vdots\, \cdots \, A^{n-1} B \right]$$

$$W_c = \left[B_c \,\vdots\, A_c B_c \,\vdots\, \cdots \, A_c^{n-1} B_c \right]$$

known

as B_c is known, and A_c is known (as characteristic polynomial of A is known)

$B_c = TB$

$A_c = TAT^{-1}$

$$W_c = \left[TB \,\vdots\, TA\underbrace{T^{-1}T}_{I}B \,\vdots\, \cdots \,\vdots\, (TAT^{-1})^{n-1} TB \right]$$

$$(TAT^{-1})^{n-1} \to \underbrace{TA\underbrace{T^{-1}T}_{I}AT^{-1} \cdots TAT^{-1}}_{(n-1)}TB$$

$$= \left[TB \,\vdots\, TAB \,\vdots\, \cdots \,\vdots\, TA^{n-1}B \right]$$

$$= T\left[B \,\vdots\, AB \,\vdots\, \cdots \,\vdots\, A^{n-1}B \right]$$

$$= TW$$

If W is invertible ($\equiv \operatorname{rank}(W) = n$)

then $\quad W_c = TW$

$$\Rightarrow W_c W^{-1} = T$$

possibly non-zero entries

$$W_c = \begin{bmatrix} 1 & -a_1 & * & & \\ 0 & 1 & * & & \\ \vdots & 0 & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & & 1 \end{bmatrix}$$

zero entries

$A_c = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \end{bmatrix}$, $B_c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$