ELL333

MULTIVARIABLE CONTROL

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$$W = \left[B \;\vdots\; AB \;\vdots\; A^2B \;\dots\; A^{n-1}B \right]$$

$$\dot{x} = Ax + Bu \quad , x(0) = x_0$$

Solution,

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)} B\, u(\tau)\, d\tau$$

Suppose $x_0 = 0$, (origin)

We know from Cayley-Hamilton Theorem that A satisfies it's own characteristic polynomial

$$\lambda^n + a_1\lambda^{n-1} + \dots a_n = 0$$

$$\Rightarrow \quad A^n + a_1 A^{n-1} + \dots + a_n I = 0$$

$$\Rightarrow \quad A^n = -a_1 A^{n-1} - a_2 A^{n-2} - \dots - a_n I$$

also $A^{n+1} = -a_1 A^n - a_2 A^{n-1} \dots - a_n A$

$\to$ is in terms of $A^{n-1}, A^{n-2}, \dots A, I$

In computing matrix exponential

$$e^{A(t-\tau)} = I + A(t-\tau) + A^2\frac{(t-\tau)^2}{2!} + \dots + A^{n-1}\frac{(t-\tau)^{n-1}}{(n-1)!}$$

$$+ A^n \frac{(t-\tau)^n}{n!} + \dots$$

all terms of power 'n' or higher can be expressed in terms of $I, A, \dots A^{n-1}$

$$= I f_0(t-\tau) + A f_1(t-\tau) + \dots + A^{n-1} f_{n-1}(t-\tau)$$

Replace this in solution,

$$x(t) = \int_0^t \left[I f_0(t-\tau) + A f_1(t-\tau) + \dots + A^{n-1} f_{n-1}(t-\tau) \right] B u(\tau) \, d\tau$$

$$= \int_0^t I f_0(t-\tau) B u(\tau) \, d\tau + \int_0^t A f_1(t-\tau) B u(\tau) \, d\tau + \dots + \int_0^t A^{n-1} f_{n-1}(t-\tau) B u(\tau) \, d\tau$$

$$= \left[\int_0^t f_0(t-\tau) u(\tau) \, d\tau \right] B + \left[\int_0^t f_1(t-\tau) u(\tau) \, d\tau \right] AB + \dots + \left[\int_0^t f_{n-1}(t-\tau) u(\tau) \, d\tau \right] A^{n-1} B$$

If, rank $[B \mid AB \mid \dots A^{n-1} B] = n$, then, $\{B, AB, \dots A^{n-1} B\}$ form a basis and $x(t)$ can be any point in state space. So, in principle, through proper choice of input $u(\tau)$, $0 \le \tau \le t$, any point in state space can be reached from the origin.

↳ definitions of controllability and reachability