ELL 333

MULTIVARIABLE CONTROL

27.10.2018 (notes)

Relation between state-space models and transfer function models:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Transfer Function $\rightarrow \dfrac{Y(s)}{U(s)} = \dfrac{\text{numerator}(s)}{\text{denominator}(s)}$ polynomials in s

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Exponential signals play an important role in linear systems theory. So, consider an input,

$$u(t) = e^{st}, \quad t \geq 0$$

Solution: $x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)} B\, u(\tau)\, d\tau$

$$= e^{At}x_0 + \int_0^t e^{A(t-\tau)} B\, e^{s\tau}\, d\tau$$

$$= e^{At}x_0 + e^{At}\int_0^t e^{-A\tau} B\, e^{s\tau}\, d\tau$$

$$= e^{At}x_0 + e^{At}\left[\int_0^t e^{(sI-A)\tau}\, d\tau\right] B$$ [may be checked from definition of matrix exponential]

$$= e^{At}x_0 + e^{At}\left[\left(e^{(sI-A)t} - I\right)(sI-A)^{-1}\right] B$$

$$= e^{At}x_0 + e^{At}\left(e^{st}.e^{-At} - I\right)(sI-A)^{-1} B$$

$$= e^{At}x_0 + \left(e^{st}.I - e^{At}\right)(sI-A)^{-1} B$$

$$\Rightarrow y(t) = Ce^{At}x_0 + \left[C\left(e^{st}.I - e^{At}\right)(sI-A)^{-1}B + De^{st}\right]$$

$$= \underbrace{Ce^{At}x_0}_{\substack{\text{initial}\\ \text{condition}\\ \text{response}}} + \underbrace{\left[\left(C(sI-A)^{-1}B + D\right)e^{st} - Ce^{At}(sI-A)^{-1}B\right]}_{\substack{\text{input response}\\ \text{(hence, the } s\text{-dependence)}}}$$

$$= \underbrace{Ce^{At}\left[x_0 - (sI-A)^{-1}B\right]}_{\text{transient response}} + \underbrace{\left[C(sI-A)^{-1}B + D\right]e^{st}}_{\text{pure exponential response}}$$

$\therefore$ Transfer function from u to y is $\rightarrow C(sI-A)^{-1}B + D$.

ROUGH: A below is not necessarily same as A above.

$$e^{At} = I + At + A^2 \frac{t^2}{2!} + \dots$$

$$\frac{d}{dt} e^{At} = A + A^2 t + A^3 \frac{t^2}{2!} + \dots$$

$$= A\left(I + At + A^2 \frac{t^2}{2!} + \dots\right) \text{ or } = \left(I + At + A^2 \frac{t^2}{2!} + \dots\right) A$$

$$= Ae^{At} \qquad \text{or} = e^{At} A$$

$$\frac{d}{dt} A^{-1} e^{At} = \frac{d}{dt} \left\{ A^{-1} + It + A \frac{t^2}{2!} + \dots \right\}$$

$$\text{or } \frac{d}{dt} e^{At} A^{-1} = I + A + A^2 \frac{t^2}{2!} + \dots$$

(if inverse exists)

$$= e^{At}$$

Above relation can also be obtained directly by taking the Laplace transform.

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

$$\Rightarrow \; sX(s) - x_0 = AX(s) + BU(s), \quad Y(s) = CX(s) + DU(s)$$

$$\Rightarrow \; (sI - A) X(s) = x_0 + BU(s)$$

$$\Rightarrow \; X(s) = (sI - A)^{-1} x_0 + (sI - A)^{-1} BU(s)$$

$$\Rightarrow \; Y(s) = \underbrace{C(sI - A)^{-1} x_0}_{\text{initial condition response}} + \underbrace{\left[C(sI - A)^{-1} B + D\right]}_{\text{input response}} U(s)$$

Transfer function from U to Y is $C(sI - A)^{-1} B + D$.

Two important parts of transfer function.

$$\frac{\text{numerator}(s)}{\text{denominator}(s)} \begin{array}{l} \leftarrow \text{zeroes are roots of this} \\ \leftarrow \text{poles are roots of this} \end{array}$$

Interpretation of poles:

The $(sI-A)^{-1}$ term in the transfer function is $\frac{\operatorname{adj}(sI-A)}{\det(sI-A)}$

Therefore, poles are roots of $\det(sI-A)=0$, and so they are eigenvalues of the A matrix.

Further, associated with each eigenvalue λ is an eigenvector v. (i.e $Av=\lambda v$, $v \neq 0$

If $x_0 = v$ and $u=0$, then $y = Ce^{At}v = Ce^{\lambda t}v$

$\left[\because e^{At}v = (I+At+\frac{A^2}{2!}t^2+\dots)v = (1+\lambda t+\frac{\lambda^2 t^2}{2!}+\dots)v = e^{\lambda t}v\right]$

Therefore, for an initial condition that is an eigen-vector, the state $(x(t)=e^{\lambda t}v)$ stays along the eigenvec--tor.

Poles depend only on the matrix A.

Interpretation of zeros:

Recall from above that

$$y(t) = \underbrace{Ce^{At}[x_0-(sI-A)^{-1}B]}_{\text{transient response}} + \underbrace{[C(sI-A)^{-1}B+D]e^{st}}_{\text{pure exponential response}}$$

So, those values of $s=s_0$ for which the 'pure exponential response' is zero $[C(s_0I-A)^{-1}B+D=0]$, are called zeroes.

Associated with this zero is a direction $x_0=(s_0I-A)^{-1}B$ such that for this initial condition, the output is identically zero.

Therefore, for $u(t) = e^{s_0 t}$ (s_0 is such that $C(s_0 I - A)^{-1}B + D = 0$) and $x_0 = (s_0 I - A)^{-1}B$, the output $y(t) = 0$.

This association of a direction $((s_0 I - A)^{-1}B)$ with the exponent $(s = s_0)$ of e^{st} in the case of zeroes is similar to the association of a direction (eigenvector) with the eigenvalues/poles.

Let us consider an example,

$$\dot{x} = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{B} u$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x \qquad (D = 0)$$

Transfer function from $u \to y$ is $C(sI - A)^{-1}B$

$$(sI - A) = \begin{bmatrix} s-1 & -1 \\ 0 & s-2 \end{bmatrix}$$

$$\Rightarrow (sI - A)^{-1} = \frac{1}{(s-1)(s-2)} \begin{bmatrix} s-2 & 1 \\ 0 & s-1 \end{bmatrix}$$

Transfer function is $\begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{(s-1)(s-2)} \begin{bmatrix} s-2 & 1 \\ 0 & s-1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$= \frac{1}{(s-1)(s-2)} \begin{bmatrix} s-2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \frac{1}{(s-1)(s-2)} \cdot (2s - 3) = \frac{2(s - 3/2)}{(s-1)(s-2)}$$

From transfer function, poles are $+1, +2$.

These are the same as eigenvalues of A.

Eigenvector for $s = 1$,

$$sI - A = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Eigenvector for $s = 2$

$$sI - A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

From transfer function zero is $3/2$.

Therefore, for input $u = e^{\frac{3}{2}t}$, $t \geq 0$ and initial condition $x_0 = (\frac{3}{2}I - A)^{-1}B$, output is zero.

$$\frac{3}{2}I - A = \begin{bmatrix} 1/2 & -1 \\ 0 & -1/2 \end{bmatrix}$$

$$\left(\frac{3}{2}I - A\right)^{-1} = \frac{1}{-1/4}\begin{bmatrix} -1/2 & 1 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 0 & -2 \end{bmatrix}$$

$$x_0 = \begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$$

— end of example.