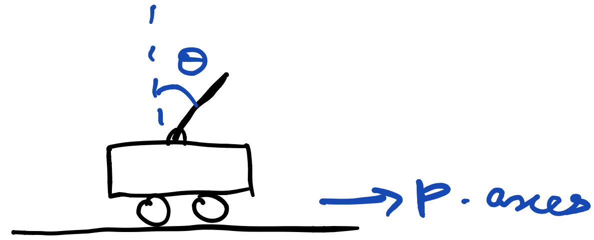
MULTIVARIABLE CONTROL

ELL 333

02.11.2018

Example: For balance system



$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & m^2 l^2 g/\mu & 0 & 0 \\ 0 & M_t m g l/\mu & 0 & 0 \end{bmatrix}, \quad x = \begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix}$$

p – position

θ – angle.

How many states need to be directly measured? → for implementing the controller

What C matrices ensure $\operatorname{rank}\left(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}\right) = n$?

This condition comes from trying to place the eigenvalues of $A-LC$, which is similar to $A-BK$, if we take transpose

- ✓ $C = I_{4\times4}$
- $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$?
- others ?
- $C = [1\ 0\ 0\ 0]$? → this should also work ?
- $C = [0\ 1\ 0\ 0]$?
- $C = [0\ 0\ 1\ 0]$?
- $C = [0\ 0\ 0\ 1]$?

Using the substitutions $A \leftrightarrow A^T = \hat{A}$, $C \leftrightarrow C^T = \hat{B}$, and $L \leftrightarrow L^T = \hat{K}$, following statements can be made:

1. For a single output system, the condition on (A, C) for the eigenvalues of $A - LC$ to be placed at desired locations is,

$$\text{rank}\left(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}\right) = n.$$

$$\left[\begin{array}{l} \text{because } \text{rank}\left([C^T \; A^TC^T \dots (A^T)^{n-1}C^T]\right) = n \\ \Rightarrow \text{rank}\left([C^T A^TC^T \dots (A^{n-1})^T C^T]\right) = n \\ \Rightarrow \text{rank}\left([C^T \; A^TC^T \dots (A^{n-1})^T C^T]^T\right) = n \\ \Rightarrow \text{rank}\left(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}\right) = n \end{array}\right.$$

If $\phi(\lambda) = 0$ is the desired characteristic polynomial, then

$$\hat{K} = [0 \; 0 \dots 1][\hat{B} \;\; \hat{A}\hat{B} \dots (\hat{A})^{n-1}\hat{B}]^{-1} \phi(\hat{A})$$

$$\Rightarrow L^T = [0 \; 0 \dots 1][C^T \; A^TC^T \dots (A^T)^{n-1}C^T]^{-1} \phi(A^T)$$

$$= [0 \; 0 \dots 1]\left[C^T A^T C^T \dots (A^{n-1})^T C^T\right]^{-1} (\phi(A))^T$$

$$= [0 \; 0 \dots 1]\left(\begin{bmatrix} C \\ CA \\ CA^{n-1} \end{bmatrix}^T\right)^{-1} (\phi(A))^T$$

$$= [0 \; 0 \dots 1]\left(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}^{-1}\right)^T (\phi(A))^T$$

$$\Rightarrow L = \phi(A) \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

2. Eigenvalue assignment of $A - LC$ (C is $1 \times n$) to desired locations through choice of L can be done for inspection for the case where

$$A = \begin{bmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ -a_n & 0 & 0 & \dots & 0 \end{bmatrix}, \quad C = [1 \; 0 \dots 0]$$

because if $L = \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \end{bmatrix}$, then

$$A - LC = \begin{bmatrix} -(a_1+l_1) & 1 & 0 & \cdots & 0 \\ -(a_2+l_2) & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ -(a_n+l_n) & 0 & 0 & \cdots & 0 \end{bmatrix},$$ whose characteristic

polynomial is $\lambda^n + (a_1+l_1)\lambda^{n-1} + \dots + (a_n+l_n) = 0$.

A linear system can be transformed into this form if

$$\text{rank}\left(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}\right) = n$$

3. Motivation for $\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$ in general (not necessary single output)

Consider the system $\dot{x} = Ax$
$y = Cx$

If <u>C is invertible</u>, x can be estimated from $C^{-1}y$.

$\rightarrow$ rank $\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$

If not, consider

$$\dot{y} = C\dot{x} = CAx$$
$$\ddot{y} = C\ddot{x} = CA^2x$$
$$\vdots$$
$$y^{(n-1)} = Cx^{(n-1)} = CA^{n-1}x$$

Taking higher derivatives gives powers of A higher than '$n-1$', and by Cayley Hamilton Theorem, they can be expressed in terms of $\{I, A, \dots A^{n-1}\}$.

$$\Rightarrow \begin{bmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n-1)} \end{bmatrix} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{(=W)} x \Rightarrow$$ If W is invertible, x can be obtained from y.

Observability: A linear system is observable if for any time t_f (assume $t_i = 0$), the state $x(t_f)$ can be obtained from knowledge of output y and input u in the duration $[0, t_f]$.

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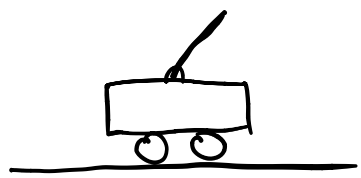

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