ELL333

MULTIVARIABLE CONTROL

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Example: Multiple inputs

$$A = \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\rightarrow \text{rank}([B \ AB]) = ?$

(# states) x (# inputs)

2×2

$$\dot{x} = Ax + Bu$$

2 states, 2 inputs

Want to design $u = -Kx$

u: 2×1, K: 2×2, x: 2×1

Here $K = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$

We have desired eigenvalues, can we, using K, place eigenvalues of $A-BK$ to these desired locations.

$$A - BK = \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$$

$$= \begin{bmatrix} -k_{11} & 2-k_{12} \\ -k_{21} & 3-k_{22} \end{bmatrix}$$

Characteristic polynomial: $\det\left(\begin{bmatrix} \lambda + k_{11} & -2+k_{12} \\ k_{21} & \lambda - 3 + k_{22} \end{bmatrix}\right)$

$$\Rightarrow (\lambda + k_{11})(\lambda - 3 + k_{22}) - k_{21}(k_{12} - 2) = 0$$

$$\Rightarrow \lambda^2 + (k_{11} + k_{22} - 3)\lambda + k_{11}(k_{22} - 3) - k_{21}(k_{12} - 2) = 0$$

Suppose desired eigenvalues $\lambda_1 = -3, \lambda_2 = -5,$ then what is K?

$\hookrightarrow$ Desired characteristic polynomial:

$$(\lambda + 3)(\lambda + 5) = 0$$

$$\Rightarrow \lambda^2 + 8\lambda + 15 = 0$$

Equating the co-efficients

$$\left.\begin{aligned} k_{11} + k_{22} - 3 &= 8 \\ k_{11}(k_{22} - 3) - k_{21}(k_{12} - 2) &= 15 \end{aligned}\right\}$$ 2 equations, 4 unknowns

$\Rightarrow$ UNDERDETERMINED

Soumyadip's choice

#1 $k_{12} = 2$, $k_{21} = $ anything, $k_{11} = 3$, $k_{22} = 8$

#2 $k_{21} = 0$, $k_{12} = $ anything, $k_{11} = 3$, $k_{22} = 8$

#3 " , " , $k_{11} = 8$, $k_{22} = 3$

$\hookrightarrow$ doesn't work

#5 $k_{21} = 5$, $k_{12} = -5$ $k_{11} = 8$, $k_{22} = 3$

#4 " , " , $k_{11} = 5$, $k_{22} = 6$

Point of example is to show how in multi-input case there may be multiple options for K.

Bicycle example code for another example of multi-input design.

Example:

$$A = \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\dot{x} = Ax + Bu$$

2 states, 2 inputs

Want to design $u = -Kx$

$\underset{1\times 1}{u}$, $\underset{1\times 2}{K}$, $\underset{2\times 1}{x}$

Here $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$ $\quad A - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} k_1 & k_2 \end{bmatrix} =$

$$A - BK = \begin{bmatrix} -k_1 & 2-k_2 \\ 0 & 3 \end{bmatrix}$$

Characteristic polynomial: $(\lambda + k_1)(\lambda - 3) = 0$
$\Rightarrow \lambda_1 = 3, \lambda_2 = -k_1$

$\text{rank}\begin{bmatrix} B & AB \end{bmatrix} = ?$

$\Rightarrow \text{rank}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = 1$ (consistent)

What about $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$?

$A - BK = ? = \begin{bmatrix} 0 & 2 \\ -k_1 & 3-k_2 \end{bmatrix}$

its characteristic polynomial $= ?$ $\lambda(\lambda + k_2 - 3) + 2k_1 = 0$

$\text{rank}([B \ \ AB]) = ?$ $\text{rank}\left(\begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}\right) = 2$