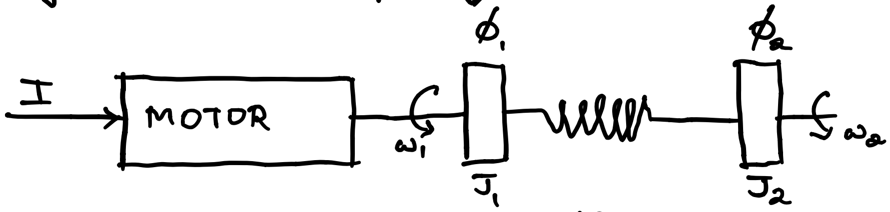
ELL 333 MINOR TEST 1 MARKS = 15 DURATION = 1 HOUR

1. Consider a system consisting of a motor driving two masses that are connected by a torsional spring as shown in the diagram below.



This system can represent a motor with a flexible shaft that drives a load. Assuming that the motor delivers a torque that is proportional to the current I, the dynamics of the system can be described by the equations,

$$J_1 \frac{d^2\phi_1}{dt^2} + c\left(\frac{d\phi_1}{dt} - \frac{d\phi_2}{dt}\right) + k(\phi_1 - \phi_2) = k_I I,$$

$$J_2 \frac{d^2\phi_2}{dt^2} + c\left(\frac{d\phi_2}{dt} - \frac{d\phi_1}{dt}\right) + k(\phi_2 - \phi_1) = T_d,$$

where ϕ_1 and ϕ_2 are the angles of the two masses, $\omega_i = d\phi_i/dt$ are their velocities, J_i represents moments of inertia, c is the damping coefficient, k represents the shaft stiffness, k_I is the torque constant for the motor, and T_d is the disturbance torque applied at the end of the shaft. Similar equations are obtained for a robot with flexible arms and for the arms of DVD and optical disk drives.

Derive a state space model for the system using the state variables $x_1 = \phi_1$, $x_2 = \phi_2$, $x_3 = \omega_1/\omega_0$, and $x_4 = \omega_2/\omega_0$, where $\omega_0 = \sqrt{\dfrac{k(J_1+J_2)}{J_1 J_2}}$.

$\boxed{5 \text{ marks}}$

ANS. $x_1 = \phi_1 \Rightarrow \dot{x}_1 = \dot{\phi}_1 = \omega_1 = \omega_0 x_3$

||ly, $x_2 = \phi_2 \Rightarrow \dot{x}_2 = \dot{\phi}_2 = \omega_2 = \omega_0 x_4$

Now, $\dot{x}_3 = \frac{1}{\omega_0}\dot{\omega}_1 = \frac{1}{\omega_0}\left[k_1 I - c(\omega_1 - \omega_2) - k(x_1 - x_2)\right] \times \frac{1}{J_1}$

$= \frac{1}{\omega_0 J_1}\left[k_1 I - c\,\omega_0 (x_3 - x_4) - k(x_1 - x_2)\right]$

and, $\dot{x}_4 = \frac{1}{\omega_0}\dot{\omega}_2 = \frac{1}{\omega_0} \times \frac{1}{J_2}\left[T_d - k(x_2 - x_1) - c\,\omega_0 (x_4 - x_3)\right]$

$$\therefore \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \omega_0 & 0 \\ 0 & 0 & 0 & \omega_0 \\ -\frac{k}{\omega_0 J_1} & \frac{k}{\omega_0 J_1} & -\frac{c}{J_1} & \frac{c}{J_1} \\ \frac{k}{\omega_0 J_2} & -\frac{k}{\omega_0 J_2} & \frac{c}{J_2} & -\frac{c}{J_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{k_1}{\omega_0 J_1} \\ 0 \end{bmatrix} I + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{\omega_0 J_2} \end{bmatrix} T_d$$

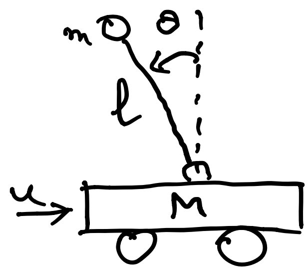
5

2. The dynamics of a simplified cart-pendulum system where the position of the base does not need to be controlled are,

5 marks

$$\frac{d}{dt}\begin{bmatrix}\theta\\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}\dot{\theta}\\ \frac{mgl}{J_t}\sin\theta - \frac{r}{J_t}\dot{\theta} + \frac{l}{J_t}u\cos\theta\end{bmatrix}, \quad y=\theta$$

Find the equilibrium points. Discuss how these change as u increases from 0 to infinity.



ANS. Equilibrium point $\Rightarrow \frac{d}{dt}\begin{bmatrix}\theta\\ \dot{\theta}\end{bmatrix}=0.$

$$\Rightarrow \dot{\theta}=0 \quad \text{and} \quad \frac{mgl}{J_t}\sin\theta - \frac{r}{J_t}\dot{\theta} + \frac{l}{J_t}u\cos\theta = 0$$

$$\Rightarrow \dot{\theta}=0 \quad \text{and} \quad \underbrace{mg\sin\theta + u\cos\theta}_{\sin(\theta+\phi)=0} = 0 \qquad 2.5$$

$$\sin(\theta+\phi)=0, \quad \tan\phi = \frac{u}{mg}$$

$$\Rightarrow \theta = n\pi - \phi, \quad n = 0, \pm 1, \ldots$$

if $u=0, \Rightarrow \theta = n\pi$

if $u = mg \Rightarrow \theta = n\pi - \pi/4$ $\qquad$ 2.5

if $u \to \infty \Rightarrow \theta = n\pi - \pi/2$

3. A simplified model of bicycle dynamics are

5 marks

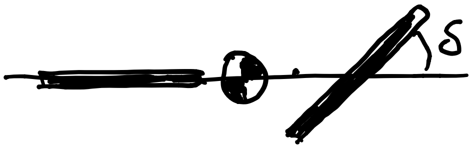
$$J\frac{d^2\phi}{dt^2} - \frac{Dv_0}{b}\frac{d\delta}{dt} = mgh\,\phi + \frac{mv_0^2 h}{b}\delta.$$

Show that these can be represented in state space form as

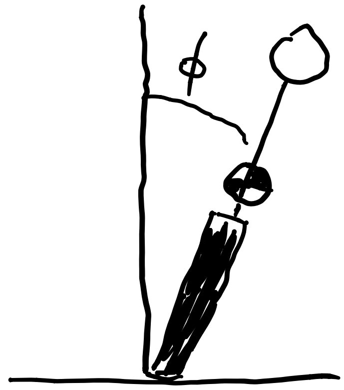
$$\frac{d}{dt}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ \frac{mgh}{J} & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}Dv_0/bJ\\ mv_0^2h/bJ\end{bmatrix}u,$$

$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix},$$

where input u is steering angle δ and output y is the tilt angle ϕ. If $u=0$ what are the eigenvalues of the 'A' matrix of the above system?



TOP VIEW



REAR VIEW

ANS. Choose $x_1 = \phi$, $x_2 = \dot{\phi} - \frac{Dv_0}{bJ}\delta$

$$\Rightarrow \dot{x}_1 = x_2 + \frac{Dv_0}{bJ}\delta \qquad 3$$

$$\Rightarrow \dot{x}_2 = \frac{1}{J}\left[mgh\,x_1 + \frac{mv_0^2 h}{b}\delta\right]$$

$$\therefore \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{mgh}{J} & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} Dv_0/bJ \\ mv_0^2 h/bJ \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

— If $u=0$, eigenvalues of $\begin{bmatrix} 0 & 1 \\ \frac{mgh}{J} & 0 \end{bmatrix}$ are $\pm\sqrt{\frac{mgh}{J}}$ 2