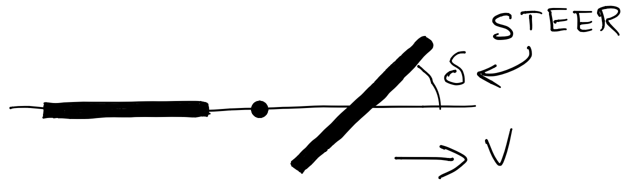
ELL 333 MINOR TEST 2 MARKS = 15 DURATION = 1 HOUR

SOLUTIONS

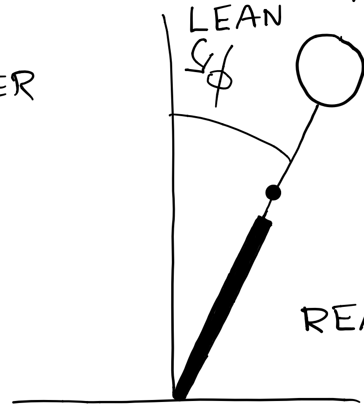
1. The "inverted pendulum model" of a bicycle is,

$$J\frac{d^2\phi}{dt^2} - mgh\,\phi = \frac{DV}{b}\frac{d\delta}{dt} + \frac{mV^2h}{b}\,\delta,$$

where ϕ is the LEAN angle, δ is the STEER angle (and an input in this model), V is the forward velocity, and $\{J, m, g, h, D, b\}$ are bicycle parameters (all are positive).



TOP VIEW



REAR VIEW

$\boxed{\text{4 MARKS}}$

(a) Determine the **stability** of the upright position.

(b) A control strategy $\delta = -k\phi$ ("STEER in opposite direction to LEAN") is designed. Find the values of k for which the upright position is asymptotically stable.

(c) What happens to the minimum value of k for asymptotic stability when forward velocity increases?

Solution.

(a) $$J\frac{d^2\phi}{dt^2} - mgh\,\phi = \frac{DV}{b}\frac{d\delta}{dt} + \frac{mV^2h}{b}\,\delta$$

V is forward velocity

$$x_1 = \phi, \quad x_2 = \dot{\phi}, \quad u = \delta$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{mgh}{J}x_1 + \frac{DV}{bJ}\frac{du}{dt} + \frac{mV^2h}{bJ}u$$

$$A = \begin{bmatrix} 0 & 1 \\ \frac{mgh}{J} & 0 \end{bmatrix} \quad \text{eigenvalues} = \pm\sqrt{\frac{mgh}{J}}$$

$\Rightarrow$ unstable

1

(b) $u = \delta = -k\phi$ (steer opposite to lean)

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{mgh}{J} x_1 - \frac{DV}{bJ} k x_2 - \frac{mV^2 h}{bJ} k x_1$$

$$A = \begin{bmatrix} 0 & 1 \\ \frac{mgh}{J} - \frac{mV^2h}{bJ}k & -\frac{DV}{bJ}k \end{bmatrix}$$

eigenvalues are roots of

$$\lambda\left(\lambda + \frac{DV}{bJ}k\right) - \left(\frac{mgh}{J} - \frac{mV^2h}{bJ}k\right) = 0$$

$$\Rightarrow \quad \lambda = \frac{1}{2}\left\{-\frac{DV}{bJ}k \pm \sqrt{\left(\frac{DV}{bJ}k\right)^2 + 4\left(\frac{mgh}{J} - \frac{mV^2h}{bJ}k\right)}\right\}$$

For asymptotic stability, $k > 0$ as $D, V, b, J > 0$

and $\frac{mgh}{J} - \frac{mV^2h}{bJ}k < 0$

$$\Rightarrow \quad k > \frac{gb}{V^2} \qquad 2$$

(c) $k_{min} = \frac{gb}{V^2}$

1

as forward velocity increases, the minimum value of k decreases. This means less control effort (in this model).

2. Consider the linear system $\dot{x} = Ax$. Find $x(t)$ for

(a) $A = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$, $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and

4 MARKS

(b) $A = \begin{bmatrix} 0 & -\omega & 1 & 0 \\ \omega & 0 & 0 & 1 \\ 0 & 0 & 0 & -\omega \\ 0 & 0 & \omega & 0 \end{bmatrix}$, $x(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$,

using direct computation of matrix exponential or any other method.

Solution: We know $x(t) = e^{At} x(0)$.

(a) Suppose $A = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$

direct computation approach $\rightarrow e^{At} = I + tA + \frac{t^2}{2!}A^2 + \cdots + \frac{t^k}{k!}A^k + \cdots$

$$A^2 = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}\begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} = \begin{bmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{bmatrix}\begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} = \begin{bmatrix} 0 & +\omega^3 \\ -\omega^3 & 0 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 0 & +\omega^3 \\ -\omega^3 & 0 \end{bmatrix}\begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} = \begin{bmatrix} \omega^4 & 0 \\ 0 & \omega^4 \end{bmatrix}$$

$$\therefore A^k = \begin{cases} \begin{bmatrix} 0 & -\omega^k \\ \omega^k & 0 \end{bmatrix} \text{ if } k = 1, 5, 9, \ldots & \begin{bmatrix} 0 & \omega^k \\ -\omega^k & 0 \end{bmatrix} \text{ if } k = 3, 7, \ldots \\ \begin{bmatrix} -\omega^k & 0 \\ 0 & -\omega^k \end{bmatrix} \text{ if } k = 2, 6, \ldots & \begin{bmatrix} \omega^k & 0 \\ 0 & \omega^k \end{bmatrix} \text{ if } k = 4, 8, \ldots \end{cases}$$

$$\Rightarrow e^{At} = \begin{bmatrix} 1-\frac{(\omega t)^2}{2!}+\frac{(\omega t)^4}{4!}+\dots & -\omega t+\frac{(\omega t)^3}{3!}\dots \\ \omega t-\frac{(\omega t)^3}{3!}+\dots & 1-\frac{(\omega t)^2}{2!}+\frac{(\omega t)^4}{4!}+\dots \end{bmatrix}$$

as $e^{j\omega t} = 1 + j\omega t + \frac{(j\omega t)^2}{2!} + \frac{(j\omega t)^3}{3!} + \frac{(j\omega t)^4}{4!} + \dots$

$\quad\ \ \Vert \qquad\quad \Downarrow$

$\cos\omega t + j\sin\omega t \Rightarrow \cos\omega t = 1-\frac{(\omega t)^2}{2!}+\frac{(\omega t)^4}{4!}\dots$

$\sin\omega t = \omega t - \frac{(\omega t)^3}{3!} + \dots$

$$\Rightarrow e^{At} = \begin{bmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{bmatrix} \Rightarrow x(t) = \begin{bmatrix} \cos\omega t \\ \sin\omega t \end{bmatrix} \qquad 2$$

(b)

$$A = \left[\begin{array}{cc|cc} \overset{\to B}{0} & -\omega & 1 & \overset{\to I_{2\times2}}{0} \\ \omega & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & -\omega \\ 0 & 0 & \omega & 0 \end{array}\right] = \begin{bmatrix} B & I \\ 0 & B \end{bmatrix}$$

$\quad\ \ \to 0_{2\times2} \qquad \to B$

$$A^2 = \begin{bmatrix} B & I \\ 0 & B \end{bmatrix}\begin{bmatrix} B & I \\ 0 & B \end{bmatrix} = \begin{bmatrix} B^2 & 2B \\ 0 & B^2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} B^2 & 2B \\ 0 & B^2 \end{bmatrix}\begin{bmatrix} B & I \\ 0 & B \end{bmatrix} = \begin{bmatrix} B^3 & 3B^2 \\ 0 & B^3 \end{bmatrix}$$

$$\Rightarrow e^{At} = \begin{bmatrix} I+tB+\frac{t^2}{2!}B^2+\dots & tI+\frac{t^2}{2!}.2B+\frac{t^3}{3!}3B^2+\dots \\ 0 & I+tB+\frac{t^2}{2!}B^2+\dots \end{bmatrix}$$

$$= \begin{bmatrix} e^{Bt} & t.e^{Bt} \\ 0 & e^{Bt} \end{bmatrix} = \begin{bmatrix} \cos\omega t & -\sin\omega t & t\cos\omega t & -t\sin\omega t \\ \sin\omega t & \cos\omega t & t\sin\omega t & t\cos\omega t \\ 0 & 0 & \cos\omega t & -\sin\omega t \\ 0 & 0 & \sin\omega t & \cos\omega t \end{bmatrix}$$

$$\Rightarrow x(t) = [-t\sin\omega t \quad t\cos\omega t \quad -\sin\omega t \quad \cos\omega t]^T \qquad 2$$

3. Consider the system $\dot{x} = Ax$, $x \in \mathbb{R}^n$. Suppose A has no eigenvalues with a strictly positive real part and one or more eigenvalues with a zero real part. Explain why $x=0$ is

- stable when the Jordan blocks corresponding to each eigenvalue with a zero real part are scalar (1X1) blocks.
- unstable when the Jordan block corresponding to at least one eigenvalue with a zero real part has size 2×2 or higher (it is not just a scalar 1×1 block).

$\boxed{4 \text{ MARKS}}$

Solution: The solution to $\dot{x} = Ax$ with initial condition $x(0)$ is

$$x(t) = e^{At}x(0)$$

0.5

Any $n\times n$ square matrix A has the Jordan decomposition $A = TJT^{-1}$, where J has the form

$$J = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & 1 & & \\ & 0 & \lambda_2 & & \\ & & & \ddots & \\ & & & & \lambda_n \end{bmatrix},$$

For first case, any eigenvalue with a zero real part has scalar (1X1) blocks, like λ_1.

For second case, there is at least one eigenvalue with a zero real part that has a Jordan block of size 2×2 or higher, like illustrated for λ_2.

$$A = TJT^{-1} \Rightarrow e^{At} = Te^{Jt}T^{-1}$$

For above J,

$$e^{Jt} = \begin{bmatrix} e^{\lambda_1 t} & & & & \\ & e^{\lambda_2 t} & te^{\lambda_2 t} & & \\ & 0 & e^{\lambda_2 t} & & \\ & & & \ddots & \\ & & & & e^{\lambda_n t} \end{bmatrix}$$

and the solution $x(t)$ is a linear combination of functions $e^{\lambda_1 t}$, $e^{\lambda_2 t}$, $te^{\lambda_2 t}$, ..., $e^{\lambda_n t}$.

↳ $t^2e^{\lambda_2 t}$... may also exist depending on the size of the Jordan block.

1.5

All terms $e^{\lambda_i t}$, $Re\{\lambda_i\} < 0$ decay to zero.

- In the first case, all terms with 't' (or higher powers of 't') are multiplied by $e^{\lambda_i t}$, $Re\{\lambda_i\} < 0$ and they also decay to zero.
1 Given these time dependences, we note that any initial condition either decays to $x=0$ or stays at a fixed distance from $x=0$. $\Rightarrow$ $x=0$ is stable.

- In the second case, there is at least one term with 't' (and higher powers of 't') that is multiplied by a
1 $e^{\lambda_i t}$, $Re\{\lambda_i\} = 0$. Due to this there will be some initial conditions close to $x=0$ which will diverge away from $x=0$ $\Rightarrow$ $x=0$ is unstable.

4. Consider the system $\dot{x} = A(t)x$, where

$$A(t) = \begin{bmatrix} -1 + \frac{3}{2}\cos^2 t & 1 - \frac{3}{2}\cos t \cdot \sin t \\ -1 - \frac{3}{2}\cos t \cdot \sin t & -1 + \frac{3}{2}\sin^2 t \end{bmatrix}.$$

$\boxed{\text{3 MARKS}}$

(a) Find the eigenvalues of $A(t)$.

(b) Show that $x(t) = \begin{bmatrix} -\epsilon e^{t/2}\cos t \\ \epsilon e^{t/2}\sin t \end{bmatrix}$, where $\epsilon > 0$ is a small positive number, is a solution of the system.

(c) Discuss the stability of $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ given the eigenvalues in part (a) and the solution in part (b).

Solution (a) $\lambda I - A = \begin{bmatrix} \lambda + 1 - \frac{3}{2}\cos^2 t & -1 + \frac{3}{2}\cos t \sin t \\ 1 + \frac{3}{2}\cos t \sin t & \lambda + 1 - \frac{3}{2}\sin^2 t \end{bmatrix}$

$$\det(\lambda I - A) = \left(\lambda + 1 - \frac{3}{2}\cos^2 t\right)\left(\lambda + 1 - \frac{3}{2}\sin^2 t\right) - \left(1 + \frac{3}{2}\cos t \sin t\right)\left(-1 + \frac{3}{2}\cos t \sin t\right)$$

$$= (\lambda+1)^2 - \frac{3}{2}(\lambda+1) + \left(\frac{3}{2}\right)^2 \sin^2 t \cos^2 t - \left(\frac{3}{2}\right)^2 \cos^2 t \sin^2 t + 1$$

$$= (\lambda+1)^2 - \frac{3}{2}(\lambda+1) + 1$$

equating this to zero $\Rightarrow$ $\lambda + 1 = \dfrac{\frac{3}{2} \pm \sqrt{\frac{9}{4} - 4}}{2}$

$$\Rightarrow \lambda = -1 + \frac{3}{4} \pm \frac{1}{4}\sqrt{9 - 16}$$

$$= \frac{-1 \pm j\sqrt{7}}{4}$$

These are the eigenvalues. 1

(b) To show this is a solution we calculate $\dot{x}$ and $A(t)x$ and show that they are equal.

<u>LHS</u>

$$\dot{x} = \begin{bmatrix} \epsilon \sin t \, e^{t/2} - \frac{\epsilon}{2} \cos t \, e^{t/2} \\ \epsilon \cos t \, e^{t/2} + \frac{\epsilon}{2} \sin t \, e^{t/2} \end{bmatrix}$$ 0.5

<u>RHS</u>

$$A(t)x = \begin{bmatrix} -1+\frac{3}{2}\cos^2 t & 1-\frac{3}{2}\cos t \sin t \\ -1-\frac{3}{2}\cos t \sin t & -1+\frac{3}{2}\sin^2 t \end{bmatrix} \begin{bmatrix} -\epsilon \cos t \, e^{t/2} \\ \epsilon \sin t \, e^{t/2} \end{bmatrix}$$

$$= \epsilon . e^{t/2} \begin{bmatrix} -1+\frac{3}{2}\cos^2 t & 1-\frac{3}{2}\cos t \sin t \\ -1-\frac{3}{2}\cos t \sin t & -1+\frac{3}{2}\sin^2 t \end{bmatrix} \begin{bmatrix} -\cos t \\ \sin t \end{bmatrix}$$

$$= \epsilon e^{t/2} \begin{bmatrix} +\cos t - \frac{3}{2}\cos^3 t + \sin t - \frac{3}{2}\cos t \sin^2 t \\ \cos t + \frac{3}{2}\cos^2 t \sin t - \sin t + \frac{3}{2}\sin^3 t \end{bmatrix}$$

$$= \epsilon e^{t/2} \begin{bmatrix} \sin t - \frac{1}{2}\cos t \\ \cos t + \frac{1}{2}\sin t \end{bmatrix}$$ 0.5

$\Rightarrow$ LHS = RHS

$\therefore$ $x(t)$ given is a solution.

(c)

<u>Stability of $x=0$</u>: Eigenvalues have negative real parts, which points to stability. However, given solution is such that any initial condition close to $x=0$ can blow up/diverge. Therefore, it is not stable. The source of this apparent paradox is the time-dependence of $A(t)$. Conclusions about stability of $\dot{x}=Ax$ from its eigenvalues rely on A being a constant matrix as that's how solution is $e^{At}x(0)$. 1