$\dot{x} = Ax, \quad x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$A = \begin{bmatrix} \lambda & \alpha \\ 0 & \mu \end{bmatrix} \quad (\lambda \neq \mu, \alpha \neq 0)$

$\lim_{\mu \to \lambda} A = \begin{bmatrix} \lambda & \alpha \\ 0 & \lambda \end{bmatrix}$ 0.5

— eigenvalues : roots of $\det(sI - A) = 0$

$sI - A = \begin{bmatrix} s-\lambda & -\alpha \\ 0 & s-\mu \end{bmatrix}$

$\det(sI-A) = (s-\lambda)(s-\mu) = 0 \Rightarrow s = \lambda, \mu$ 0.5

$s = \lambda, \lambda$ 0.5

— eigenvectors:

for $s=\lambda$, $sI - A = \begin{bmatrix} 0 & -\alpha \\ 0 & \lambda-\mu \end{bmatrix}$

$\Rightarrow$ eigenvector is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ 0.5 $\quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ 0.5

for $s = \mu$, $sI - A = \begin{bmatrix} \mu-\lambda & -\alpha \\ 0 & 0 \end{bmatrix}$

$\Rightarrow$ eigenvector is $\begin{bmatrix} 1 \\ \frac{\mu-\lambda}{\alpha} \end{bmatrix}$ 0.5

same as above i.e only one eigenvector

— $T = \begin{bmatrix} 1 & 1 \\ 0 & \frac{\mu-\lambda}{\alpha} \end{bmatrix}$ is such that $AT = TD$, where

$D = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$. $\quad T^{-1} = \frac{\alpha}{\mu-\lambda} \begin{bmatrix} \frac{\mu-\lambda}{\alpha} & -1 \\ 0 & 1 \end{bmatrix}$

— $e^{At} = T e^{Dt} T^{-1}$

$= \begin{bmatrix} 1 & 1 \\ 0 & \frac{\mu-\lambda}{\alpha} \end{bmatrix} \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{bmatrix} \begin{bmatrix} \frac{\mu-\lambda}{\alpha} & -1 \\ 0 & 1 \end{bmatrix} \times \frac{\alpha}{\mu-\lambda}$

$= \begin{bmatrix} e^{\lambda t} & e^{\mu t} \\ 0 & \frac{\mu-\lambda}{\alpha} e^{\mu t} \end{bmatrix} \begin{bmatrix} 1 & -\frac{\alpha}{\mu-\lambda} \\ 0 & \frac{\alpha}{\mu-\lambda} \end{bmatrix}$

2 $\leftarrow$ $= \begin{bmatrix} e^{\lambda t} & \frac{\alpha}{\mu-\lambda}(e^{\mu t} - e^{\lambda t}) \\ 0 & e^{\mu t} \end{bmatrix}$

$\begin{bmatrix} e^{\lambda t} & \alpha t e^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}$ 1

— $e^{At} x(0) = \begin{bmatrix} e^{\lambda t} \\ \alpha \cdot \frac{e^{\mu t} - e^{\lambda t}}{\mu - \lambda} \end{bmatrix}$ 1

as $\lim_{\mu \to \lambda} \frac{e^{\mu t} - e^{\lambda t}}{\mu - \lambda}$ is $\frac{0}{0}$ form, apply L'Hôpital's rule.

ELL333 QUIZ 2 MARKS = 7 DURATION = 20 minutes

Q. Consider the equation $\dot{x} = Ax$, $x(0)$ where

$$A = \begin{bmatrix} \lambda & \alpha \\ 0 & \mu \end{bmatrix}, \ (\lambda \neq \mu, \alpha \neq 0), \quad x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

a) Find eigenvalues and eigenvectors of A, matrix exponential e^{At}, and using this the solution of above equation.

b) Take the limit as $\mu \to \lambda$ ($\lim_{\mu \to \lambda}$) of A, its eigenvalues, eigenvectors, and matrix exponential obtained above.

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Note:

For the matrix $B = \begin{bmatrix} \lambda & \alpha \\ 0 & \lambda \end{bmatrix}$,

eigenvalues are λ, λ (repeated)

eigenvectors: $\lambda I - B = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$e^{Bt} = I + Bt + B^2 \frac{t^2}{2!} + \dots + B^k \frac{t^k}{k!} + \dots$$

$$B^2 = \begin{bmatrix} \lambda & \alpha \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} \lambda & \alpha \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} \lambda^2 & 2\alpha\lambda \\ 0 & \lambda^2 \end{bmatrix}$$

$$B^3 = \begin{bmatrix} \lambda & \alpha \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} \lambda^2 & 2\alpha\lambda \\ 0 & \lambda^2 \end{bmatrix} = \begin{bmatrix} \lambda^3 & 3\alpha\lambda^2 \\ 0 & \lambda^3 \end{bmatrix}$$

$$B^4 = \begin{bmatrix} \lambda & \alpha \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} \lambda^3 & 3\alpha\lambda^2 \\ 0 & \lambda^3 \end{bmatrix} = \begin{bmatrix} \lambda^4 & 4\alpha\lambda^3 \\ 0 & \lambda^4 \end{bmatrix}$$

$$B^k = \begin{bmatrix} \lambda^k & k\alpha\lambda^{k-1} \\ 0 & \lambda^k \end{bmatrix} \Rightarrow e^{Bt} = \begin{bmatrix} e^{\lambda t} & \alpha t e^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}$$

$\therefore$ these quantities match the limits.