ELL 333 REMINOR TEST MARKS = 15 DURATION = 1 HOUR

SOLUTIONS

1. Is the following system completely controllable and completely observable?

[2 MARKS]

$$\dot{x} = \begin{bmatrix} -3/4 & -1/4 \\ -1/2 & -1/2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 4 & 2 \end{bmatrix} x(t)$$

Ans. $\text{rank}\left\{ \begin{bmatrix} B & AB \end{bmatrix} \right\} = ?$

$\begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$, whose determinant = 0 (1)

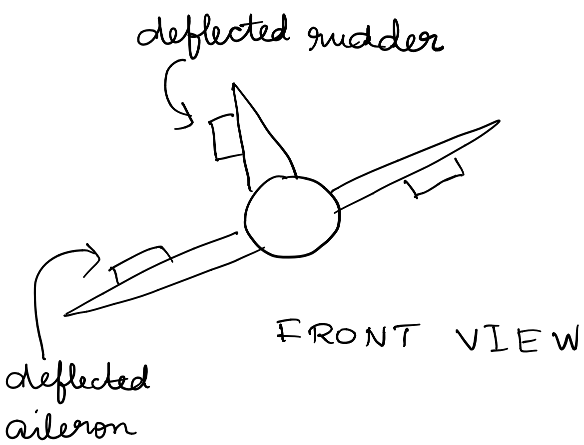
$\Rightarrow$ not completely controllable

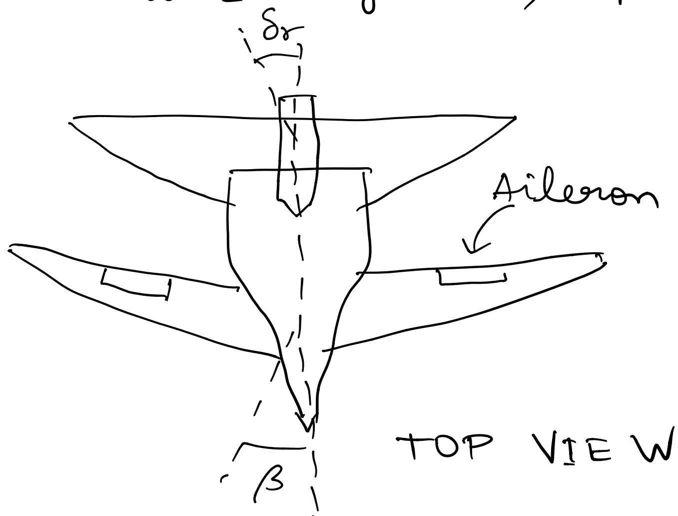
$\text{rank}\left\{ \begin{bmatrix} C \\ CA \end{bmatrix} \right\} = ?$

$\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ -4 & -2 \end{bmatrix}$, whose determinant = 0 (1)

$\Rightarrow$ not completely observable

2. An approximate linear model of the lateral dynamics of an aircraft, for a particular set of flight conditions has the state and control vectors in the perturbation quantities $x = [p \; r \; \beta \; \phi]^T$ and $u = [\delta_a \; \delta_r]^T$ where p and r are roll and yaw rates, β is an incremental sideslip angle, and ϕ is an incremental roll angle. The control inputs are incremental change in the aileron angle δ_a and in the rudder angle δ_r, respectively.





In a consistent set of units, this linearized model has, $A = \begin{bmatrix} -10 & 0 & -10 & 0 \\ 0 & -0.7 & 9 & 0 \\ 0 & -1 & -0.7 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 20 & 2.8 \\ 0 & -3.13 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

(a) Suppose a malfunction prevents manipulation of the input δ_r. Is it still possible to control the aircraft using δ_a.

(b) If the only output is a measurement of roll rate p (provided by a rate gyro), is the system observable? $\boxed{5 \text{ MARKS}}$

A(a) Let b_1 be first column of B. Then we have to check the rank of $[b_1 \; Ab_1 \; A^2b_1 \; A^3b_1] =$

$$= \begin{bmatrix} 20 & -200 & +200 & . \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 20 & -200 & 0 \end{bmatrix}$$, which is 2 $\Rightarrow$ not controllable. 2.5

(b) Here $C = [1 \; 0 \; 0 \; 0]$. We have to check rank of $\begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -10 & 0 & -10 & 0 \\ 100 & 10 & 107 & 0 \\ -1000 & -114 & (\neq 0) & 0 \end{bmatrix}$, which is 3 $\Rightarrow$ not observable. 2.5

3. Consider a simple two-dimensional system of the form $\frac{dx}{dt} = \underbrace{\begin{bmatrix} \alpha & \omega \\ -\omega & \alpha \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u$. Find the transformation (T) that converts the system into the controllable canonical form, $\dot{z} = \tilde{A}z + \tilde{B}u$, where $z = Tx$. $\boxed{4 \text{ MARKS}}$

A. In canonical form, $\tilde{A} = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix}$, $\tilde{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ 1

$\Rightarrow$ Characteristic polynomial: $\lambda^2 + a_1\lambda + a_2 = 0$

Characteristic polynomial of A: $(\lambda-\alpha)^2 + \omega^2 = 0$ 1

$\Rightarrow a_2 = \alpha^2 + \omega^2, \quad a_1 = -2\alpha$

$z = Tx \Rightarrow \dot{z} = T\dot{x} = T(Ax + Bu)$

$$= \underbrace{TAT^{-1}}_{\tilde{A}} z + \underbrace{TB}_{\tilde{B}} u$$

controllability matrix

$$\tilde{W} = \begin{bmatrix} 1 & -a_1 \\ 0 & 1 \end{bmatrix}$$

But $\tilde{W} = [\tilde{B} \quad \tilde{A}\tilde{B}] = [TB \quad TAT^{-1}TB]$

$$= [TB \quad TAB] = T[B \quad AB]$$

$$= T \underbrace{\begin{bmatrix} 0 & \omega \\ 1 & \alpha \end{bmatrix}}_{W} \Rightarrow T = \tilde{W} W^{-1}$$

$$\therefore T = \begin{bmatrix} 1 & -a_1 \\ 0 & 1 \end{bmatrix} \cdot \left(\frac{-1}{\omega}\right) \begin{bmatrix} \alpha & -\omega \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha/\omega & 1 \\ 1/\omega & 0 \end{bmatrix}$$

2

4. Prove that $e^{(N+M)t} = e^{Nt} \cdot e^{Mt}$ if $NM = MN$. Use the following steps.

$\boxed{4 \text{ MARKS}}$

(a) Show that e^{Nt} and M commute.

(b) by taking derivatives show that the derivative of $Q(t) = e^{Nt} \cdot e^{Mt}$ satisfies $\dot{Q}(t) = (N+M)\, e^{Nt} \cdot e^{Mt} = (N+M)Q(t)$

(c) Show that $Q(t) = e^{(N+M)t}$ also satisfies $\dot{Q}(t) = (N+M)Q(t)$, with $Q(0) = I$.

As $\dot{Q}(t) = (N+M)Q(t)$, $Q(0) = I$ has unique solution, this implies $e^{Nt} \cdot e^{Mt} = e^{(N+M)t}$. For $t = 1$, $e^{N} \cdot e^{M} = e^{N+M}$

(a) Follows from definition, and $NM = MN$
of e^{Nt} (2)

(b) $\dot{Q}(t) = e^{Nt} \cdot (M e^{Mt}) + (N e^{Nt}) e^{Mt}$ (1)

$= M e^{Nt} e^{Mt} + N e^{Nt} \cdot e^{Mt}$

$[\because (a)]$

$= (N+M) \, Q(t)$

(c) $\dot{Q}(t) = (N+M) \, e^{(N+M)t}$ and $Q(0) = I$. (1)