ELL 301

10.04.2019

Least Squares.

Data : $x_1, x_2, \ldots x_N$

$y_1, y_2, \ldots y_N$

what is inside this? unmodelled, ignored dynamics, disturbances, errors . .

Model : $y = Kx + n$

Square

Method : $\underbrace{\text{Minimize}}_{\text{Least}}$ cost $\sum_{i=1}^{N} (y_i - Kx_i)^2$ as a function of K.

$$\Rightarrow \hat{K} = \frac{\sum_{i=1}^{N} y_i x_i}{\sum_{i=1}^{N} x_i^2}$$

$\hat{K}$ is estimator of K

$$E(\hat{K}) = E\left\{ \frac{\sum_{i=1}^{N} y_i x_i}{\sum_{i=1}^{N} x_i^2} \right\}$$

based on model

$$= \frac{1}{\sum_{i=1}^{N} x_i^2} E\left\{ \sum_{i=1}^{N} (Kx_i + n_i) x_i \right\}$$

$$= \frac{1}{\sum_{i=1}^{N} x_i^2} E\left\{ \sum_{i=1}^{N} (Kx_i^2 + n_i x_i) \right\}$$

$$= \frac{1}{\sum_{i=1}^{N} x_i^2} \left[E\left\{ \sum_{i=1}^{N} Kx_i^2 \right\} + E\left\{ \sum_{i=1}^{N} n_i x_i \right\} \right]$$

$$= K + \frac{1}{\sum_{i=1}^{N} x_i^2} \sum_{i=1}^{N} x_i E\{n_i\}$$

$= K$, if $E\{n_i\} = 0$, noise is zero mean

$\therefore \hat{K}$ is unbiased estimator if noise is zero mean.

$$E\{(\hat{K}-K)^2\} = ? \qquad \frac{\text{variance of noise}}{\sum_{i=1}^{N} x_i^2}$$

assuming $E\{n_i\}=0$

and n_i, n_j are independent

$$\sim \frac{\text{variance of noise}}{N \cdot \underbrace{\text{variance of } x}_{\text{power}}}$$

increase N, lower variance

Is this the smallest variance that can be achieved?

Is there another method that can give an estimate of K with lower variance than this?

Cramer-Rao Bound

For an unbiased estimator $\hat{\theta}$ of θ

$\hat{\theta} \to \hat{K}$, $\theta \to K$

$$E\{(\hat{\theta}-\theta)^2\} \geq \frac{-1}{E\left\{\frac{\partial^2 \log L(y|\theta)}{\partial\theta^2}\right\}}$$

$L(y|\theta)$: likelihood

$\log L(y|\theta)$: log-likelihood

- This gives the lower bound to variance of the unbiased estimator $\hat{\theta}$
- We want to calculate this bound for a zero-mean Gaussian distribution
- and show that this is the same as that obtained from Least Squares method

Hence, least squares is the best method for above model

$$n \sim \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-\frac{n^2}{2\sigma_n^2}}$$

$$\underset{\{y_1, y_2, \dots y_N\}}{L(y|K)} = \left(\frac{1}{\sqrt{2\pi\sigma_n^2}}\right)^N \cdot e^{-\frac{\sum_{i=1}^{N} n_i^2}{2\sigma_n^2}}$$

$n_i \rightarrow (y_i - Kx_i)$

$$\frac{\partial^2}{\partial K^2} \log L(y|K)$$

$$\log L(y|K) = \log(\ ) - \frac{1}{2\sigma_n^2}\sum_{i=1}^{N}(y_i - Kx_i)^2$$

$$\frac{\partial}{\partial K}\log L(y|K) = \left(\frac{-1}{2\sigma_n^2}\right)\sum_{i=1}^{N}(-2x_i)(y_i - Kx_i)$$

$$\frac{\partial^2}{\partial K^2}\log L(y|K) = \left(\frac{-1}{\cancel{2}\sigma_n^2}\right)\sum_{i=1}^{N}\cancel{2}x_i^2$$

Lowest variance possible : $\dfrac{-1}{-\frac{1}{\sigma_n^2}\sum_{i=1}^{N} x_i^2} = \dfrac{\sigma_n^2}{\sum_{i=1}^{N} x_i^2}$,

which is the same as obtained with the least squares method for zero mean, independent noise