ELL333

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$$\dot{x} = Ax, \longrightarrow \boxed{x(t) = \underbrace{e^{At}}_{\text{matrix exponential}} \cdot c}$$

$$e^{At} = I + tA + \frac{t^2}{2!}A^2 + \dots$$

$$A = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}, \quad e^{At} = ?$$

$\dot{x} = Ax, \; x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\dot{x}_1 = \omega x_2$

$\dot{x}_2 = -\omega x_1$

$\ddot{x}_2 + \omega^2 x_2 = 0 \Leftarrow \; \Rightarrow \ddot{x}_2 = -\omega \dot{x}_1 = -\omega^2 x_2$

$$e^{At} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \frac{t^4}{4!}A^4 + \dots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t\begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} + \frac{t^2}{2!}\begin{bmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{bmatrix}$$

$$+ \frac{t^3}{3!}\begin{bmatrix} 0 & -\omega^3 \\ \omega^3 & 0 \end{bmatrix} + \frac{t^4}{4!}\begin{bmatrix} \omega^4 & 0 \\ 0 & \omega^4 \end{bmatrix} + \dots$$

$$= \begin{bmatrix} 1 - \frac{t^2}{2!}\omega^2 + \frac{t^4}{4!}\omega^4 + \dots & \omega - \frac{t^3}{3!}\omega^3 + \dots \\ -\omega + \frac{t^3}{3!}\omega^3 + \dots & 1 - \frac{t^2}{2!}\omega^2 + \frac{t^4}{4!}\omega^4 + \dots \end{bmatrix}$$

$$= \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix}$$

Direct way to compute matrix exponent-ial.

Other methods are there

$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad e^{At} = ?$$

$$e^{At} = I + At + \frac{t^2}{2!}A^2 + \cdots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} + \frac{t^2}{2!}\begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} 1 + t\lambda_1 + \frac{t^2}{2!}\lambda_1^2 + \cdots & 0 \\ 0 & 1 + t\lambda_2 + \frac{t^2}{2!}\lambda_2^2 + \cdots \end{bmatrix}$$

$$= \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$$

This is useful, because?

Given a matrix A, we can write

only possible if eigenvalues of A are distinct

?

$$A = U \underbrace{D}_{\text{diagonal}} U^{-1}$$

?

$$e^{At} = U e^{Dt} U^{-1}$$

$$e^{At} = I + tA + \frac{t^2}{2!}A^2 + \cdots$$

$$= \underset{\substack{\downarrow \\ UU^{-1}}}{I} + tUDU^{-1} + \frac{t^2}{2!} UD\underbrace{U^{-1}U}_{I}DU^{-1} + \cdots$$

$$= U\left[I + tD + \frac{t^2}{2!}D^2 + \cdots\right]U^{-1}$$

$$= Ue^{Dt}U^{-1}$$

$$A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \quad e^{At} = ?$$

$$e^{At} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix} \quad \text{(check)}$$

A square matrix A with n distinct eigenvalues, $\lambda_1, \lambda_2 \ldots \lambda_n$ and corresponding eigenvectors, $v_1, v_2 \ldots v_n$

$$Av_1 = \lambda_1 v_1$$
$$Av_2 = \lambda_2 v_2$$
$$\vdots$$
$$Av_n = \lambda_n v_n$$

'D' Diagonal matrix

$$\Rightarrow A\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}_{n\times n} = \overbrace{\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}}^{U} \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix}$$

$$\updownarrow \qquad\qquad\qquad\qquad \updownarrow$$

$$[Av_1 \;\; Av_2 \;\ldots\; Av_n] \qquad [\lambda_1 v_1 \;\; \lambda_2 v_2 \;\ldots\; \lambda_n v_n]$$

$$\Rightarrow \quad AU = UD$$
$$\Rightarrow \quad A = UDU^{-1}$$

Why is it a transformation?

$\dot{x} = Ax$

$z = Tx$, T is invertible

$\Rightarrow x = T^{-1}z$

$$\dot{z} = T\dot{x} = TAx = TAT^{-1}z$$

$$\Rightarrow \dot{z} = \underbrace{(TAT^{-1})}_{D} z$$

$z = Tx$

Mapping from one basis to another

this may lose physical identification of states