ELL 333
01.08.2019

$\dot{x} = Ax$, solution $x(t) = e^{At}x(0)$
& how to calculate e^{At}?

Why?
→ Transient response / Dynamics
→ long term behaviour / stability

If A has distinct eigenvalues, $\lambda_1, \lambda_2 \ldots \lambda_n$
we can compute e^{At} by diagonalization
and given initial condition, we can
get dynamics.
↳ in terms of $e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots e^{\lambda_n t}$

for linear systems, we talk about origin

Stability (of the equilibrium point)

If $Re\{\lambda_i\} < 0 \; \forall i$,
then the system
is stable, because
all transient terms
decay to zero.

$\dot{x} = Ax$, what are equilibrium points?
$\Rightarrow \dot{x} = 0$
$Ax = 0$
$x = 0$, $\in$ nullspace of A

fixed point or steady state

Example:

$\dot{x} = Ax, \quad x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

If we take $\lim\limits_{\lambda \to \mu}$

$A = \begin{bmatrix} \lambda & 1 \\ 0 & \mu \end{bmatrix}, \quad \lambda \neq \mu.$

Jordan Canonical Form $\begin{bmatrix} \mu & 1 \\ 0 & \mu \end{bmatrix}$

$e^{At} = \begin{bmatrix} e^{\mu t} & te^{\mu t} \\ 0 & e^{\mu t} \end{bmatrix}$

additional contribution

Eigenvalues ?

Eigenvectors ?

Diagonalization ?

$e^{At} = ?$

Eigenvalues: $\det(sI - A) = 0$

$$sI - A = \begin{bmatrix} s-\lambda & -1 \\ 0 & s-\mu \end{bmatrix}$$

$\det(sI - A) = 0 \Rightarrow (s-\mu)(s-\lambda) = 0$

$\Rightarrow s = \lambda, \mu$ $\quad \lambda, \lambda$

$\underline{s = \lambda}$ $\quad \lambda I - A = \begin{bmatrix} 0 & -1 \\ 0 & \lambda - \mu \end{bmatrix} \to \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow$ eigenvector should have $x_2 = 0$

corresponding eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\underline{s = \mu}$ $\quad \mu I - A = \begin{bmatrix} \mu - \lambda & -1 \\ 0 & 0 \end{bmatrix}$

eigenvector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ should have $(\mu - \lambda)x_1 - x_2 = 0$

corresponding eigenvector $\rightarrow \begin{bmatrix} 1 \\ \mu-\lambda \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

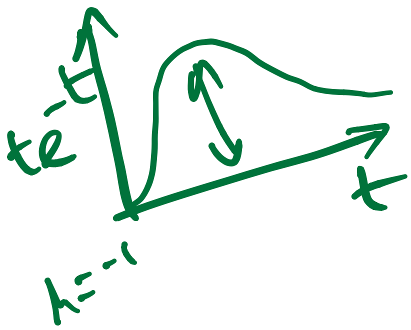
$$AU = UD, \quad U = \begin{bmatrix} 1 & 1 \\ 0 & \mu-\lambda \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A = UDU^{-1} \Rightarrow e^{At} = U e^{Dt} U^{-1}$$

$$D = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} \Rightarrow e^{Dt} = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{bmatrix}$$

$$\Rightarrow e^{At} = \frac{1}{\mu-\lambda} \begin{bmatrix} (\mu-\lambda)e^{\lambda t} & e^{\mu t} - e^{\lambda t} \\ 0 & (\mu-\lambda)e^{\mu t} \end{bmatrix}$$

$$= \begin{bmatrix} e^{\lambda t} & \dfrac{e^{\mu t} - e^{\lambda t}}{\mu-\lambda} \\ 0 & e^{\mu t} \end{bmatrix} \xrightarrow{\lim_{\lambda\to\mu}} \begin{bmatrix} e^{\mu t} & te^{\mu t} \\ 0 & e^{\mu t} \end{bmatrix}$$



$$x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow x(t) = \begin{bmatrix} e^{\lambda t} + \dfrac{e^{\mu t} - e^{\lambda t}}{\mu-\lambda} \\ e^{\mu t} \end{bmatrix}$$

As can be seen, stability / long term behaviour is still determined by sign of real part of eigenvalues.

$\therefore$ For any $\dot{x} = Ax$, stable if $\mathrm{Re}\{\text{eigenvalues of } A\} < 0$.

Quiz 1.

$$\dot{x} = Ax, \quad x(0) = \begin{bmatrix} 10^{-6} \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & \omega \\ \omega & 0 \end{bmatrix}$$

eigenvalues = ?

eigenvectors = ?

$e^{At} = ?$

$x(t) = ?$