ELL333
02.08.2019

Generally, we have input and output, in addition to the state

$$\boxed{\dot{x} = Ax} + Bu$$

Dynamics & Stability

$$y = Cx + Du$$

What does input (u) and output (y) have use for?

- to change state
- to control state

$\underbrace{}_{\text{using 'u'}}$

- to observe state

$\underbrace{}_{\text{using y}}$

Example from last time's major test

$$z = Tx \Rightarrow x = T^{-1}z$$

$$\dot{z} = T\dot{x}$$

$$= T(Ax + Bu)$$

$$= TAx + TBu$$

$$\Rightarrow \dot{z} = TAT^{-1}z + TBu$$
$$y = CT^{-1}z$$

$$\Rightarrow \dot{z} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} z + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} z$$

$$\dot{z}_1 = -z_1 + u$$
$$\dot{z}_2 = -2z_2$$
$$\dot{z}_3 = -3z_3 + u$$
$$\dot{z}_4 = -4z_4$$

$$y = z_1 + z_2$$

an instance of Kalman decomposition

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u cannot be used to change dynamics of z_2 and z_4. $\rightarrow$ "uncontrollable"

y cannot be used to observe what happens to z_3 and z_4. $\rightarrow$ "unobservable"

Convert to transfer function

$$\dot{z} = Az + Bu$$

$\Rightarrow$ $sI\,z(s) = A\,z(s) + B\,U(s)$

$\Rightarrow$ $z(s) = (sI - A)^{-1} B\, U(s)$

& $y = Cz$

$\Rightarrow$ $Y(s) = C\, z(s) = C(sI - A)^{-1} B\, U(s)$

$\Rightarrow$ Transfer function $= C\,(sI - A)^{-1} B$

$$= \frac{1}{s+1} \quad ?$$

ELL333 MAJOR TEST MARKS=35 TIME=2 HOURS

SOLUTIONS

1. Consider the system $\ddot{y} - a^2 y = u$, $a>0$, where u is the input and y is the output, both scalars. $\boxed{9 \text{ marks}}$

(a) Obtain a state-space model using the states $x_1 = y$, $x_2 = \dot{y}$.

(b) Investigate the stability of the system.

(c) Show that the model obtained in (a) is controllable.

(d) Design a control law $u = -Kx$ to place the eigenvalues at $-a, -a$. ← Specification

(e) Show that the model obtained in (a) is observable.

(f) Design an observer to estimate the state from output measurements. Describe the overall controller.

(g) Suppose instead of state estimation and state feedback, direct output feedback, $u = -ky$ is used. Is the specification given in (d) achievable? Explain.

Ans. (a)

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ a^2 & 0 \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u$$ 0.5

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(b) eigenvalues of A are $\pm a$, one of which is in RHP

$\Rightarrow$ unstable. 0.5

(c) $W_c = [B \;\; AB] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \text{rank}(W_c) = 2$ 0.5

$\Rightarrow$ controllable.

(d) When $u = -Kx = -[k_1 \;\; k_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$,

$$\dot{x} = Ax - BKx = (A-BK)x$$

$$A - BK = \begin{bmatrix} 0 & 1 \\ a^2 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}[k_1 \;\; k_2] = \begin{bmatrix} 0 & 1 \\ a^2 - k_1 & -k_2 \end{bmatrix}$$

eigenvalues of $A-BK$ are roots of $\det(\lambda I-(A-BK))=0$

$\Rightarrow \lambda(\lambda+k_2)-(a^2-k_1)=0$

$\Rightarrow \lambda^2+k_2\lambda+k_1-a^2=0$

For these to be at $-a,-a$, the gains k_1, k_2 should be chosen so that the characteristic polynomial is

$(\lambda+a)^2=0 \Rightarrow \lambda^2+2a\lambda+a^2=0$

2

$\Rightarrow k_1=2a^2,\ k_2=2a$

(e) $W_o=\begin{bmatrix} C \\ CA \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \text{rank}(W_o)=2$

0.5

$\Rightarrow$ observable.

(f) Observer is $\dot{\hat{x}}=A\hat{x}+Bu+L(y-\hat{y})$
$\hat{y}=C\hat{x}$

$\tilde{x}=x-\hat{x} \Rightarrow \dot{\tilde{x}}=A\tilde{x}-LC\tilde{x}=(A-LC)\tilde{x}$

So eigenvalues of $A-LC$ should be in LHP.

$A-LC=\begin{bmatrix} 0 & 1 \\ a^2 & 0 \end{bmatrix}-\begin{bmatrix} l_1 \\ l_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}=\begin{bmatrix} -l_1 & 1 \\ a^2-l_2 & 0 \end{bmatrix}$

eigenvalues are roots of $\lambda(\lambda+l_1)-(a^2-l_2)=0$

$\Rightarrow \lambda^2+l_1\lambda+l_2-a^2=0 \Rightarrow l_1>0,\ l_2>a^2$

(This is minimum requirement, other considerations may apply).

For the plant $\dot{x}=Ax+Bu$, $y=Cx$, the overall controller is

$$\dot{\hat{x}}=A\hat{x}+Bu+L(y-\hat{y})$$
$$\hat{y}=C\hat{x}$$
$$u=-K\hat{x}$$

3

"it is hard to imagine that the observer designed for a known input can serve to estimate the state of the process for the purpose of generating the control input." – B. Friedland in 'Control System Methods'

(g) Suppose $u = -ky = -kCx$

Then $\dot{x} = Ax + Bu = Ax - BkCx = (A - kBC)x$

$$A - kBC = \begin{bmatrix} 0 & 1 \\ a^2 & 0 \end{bmatrix} - k \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ a^2 & 0 \end{bmatrix} - k \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a^2 - k & 0 \end{bmatrix}$$

eigenvalues are roots of $\lambda^2 - (a^2 - k) = 0$

Therefore, specifications cannot be met.

2

Contrast with other parts above!

2. According to the Popov-Belevitch-Hautus test,

$\text{rank}\{[B\ AB \dots A^{n-1}B]\} = n \iff \text{rank}\{[\lambda I - A\ \ B]\} = n$ $\forall$ complex numbers λ.

A is $n \times n$ matrix

B is $n \times m$ matrix

Verify this for

(a) $A = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\boxed{8 \text{ marks}}$

(c) $A = \begin{bmatrix} -1 & -2 & -3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Ans. (a) $[B\ AB \dots A^{n-1}B] = \begin{bmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \end{bmatrix}$

$\det \begin{bmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \end{bmatrix} = (a_1 - a_2)(a_2 - a_3)(a_3 - a_1)$, which is non-zero if a_1, a_2, a_3 are distinct.

$[\lambda I - A\ \ B] = \begin{bmatrix} \lambda - a_1 & 0 & 0 & 1 \\ 0 & \lambda - a_2 & 0 & 1 \\ 0 & 0 & \lambda - a_3 & 1 \end{bmatrix}$

- rows are linearly independent if $\lambda \neq a_1, \lambda \neq a_2, \lambda \neq a_3$.
- even when $\lambda = a_1$ or $\lambda = a_2$ or $\lambda = a_3$, rows are linearly independent.
- only if $a_1 = a_2$ and $\lambda = a_1$, the rank < 3.
- same if $a_1 = a_2 = a_3$ and $\lambda = a_1$.

This is same condition as obtained above. 3

(b) $[B \ \ AB] = \begin{bmatrix} 0 & 1 \\ 1 & a \end{bmatrix}$, whose rank is 2 ← same

$[\lambda I - A \ \ B] = \begin{bmatrix} \lambda - a & -1 & 0 \\ 0 & \lambda - a & 1 \end{bmatrix}$, whose rank is 2 always. ②

(c) $[B \ \ AB \ \ A^2B \ \ A^3B] = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ whose rank is 3.

$$[\lambda I - A \ \ B] = \begin{bmatrix} \lambda+1 & 2 & 3 & 1 \\ -1 & \lambda & 0 & 0 \\ 0 & -1 & \lambda & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 2+\lambda(\lambda+1) & 3 & 1 \\ -1 & \lambda & 0 & 0 \\ 0 & -1 & \lambda & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 0 & 3+\lambda(2+\lambda(\lambda+1)) & 1 \\ -1 & 0 & \lambda^2 & 0 \\ 0 & -1 & \lambda & 0 \end{bmatrix}$$, whose rank is 3.

Again, same condition as above. ③

3. Suppose $\dot{x} = Ax + Bu$ and an invertible co-ordinate transformation $z = Tx$ is made to obtain $\dot{z} = \tilde{A}z + \tilde{B}u$.

(a) Find $\tilde{A}$ and $\tilde{B}$. $\boxed{\text{5 marks}}$

(b) Show that $\text{rank}\{[\tilde{B}\ \tilde{A}\tilde{B} \dots \tilde{A}^{n-1}\tilde{B}]\} = \text{rank}\{[B\ AB \dots A^{n-1}B]\}$.

(c) Suppose $\tilde{A} = \begin{array}{c} \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} \begin{array}{l} \updownarrow r \\ \updownarrow n-r \end{array} \\ \begin{array}{cc} \overleftrightarrow{r} & \overleftrightarrow{n-r} \end{array} \end{array}$, $\tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix} \begin{array}{l} \updownarrow r \\ \updownarrow n-r \end{array}$ and

$\text{rank}\{[\tilde{B}_1\ \ \tilde{A}_{11}\tilde{B}_1\ \dots\ \tilde{A}_{11}^{r-1}\tilde{B}_1]\} = r$. What is $\text{rank}\{[B\ AB \dots A^{n-1}B]\}$?

Ans. (a) $z = Tx \Rightarrow \dot{z} = T\dot{x} = TAx + TBu$

$$\Rightarrow \dot{z} = \underbrace{TAT^{-1}}_{\tilde{A}} z + \underbrace{TB}_{\tilde{B}} u$$ 1

(b) $\text{rank}\{[\tilde{B}\ \tilde{A}\tilde{B} \dots \tilde{A}^{n-1}\tilde{B}]\} = ?$

$[\tilde{B}\ \tilde{A}\tilde{B} \dots \tilde{A}^{n-1}\tilde{B}]$

$= [TB\ \ TAT^{-1}TB\ \dots\ \underbrace{TAT^{-1}TAT^{-1}\dots TAT^{-1}}_{n-1}TB]$

$= [TB\ \ TAB \dots TA^{n-1}B] = T\,[B\ \ AB \dots A^{n-1}B]$

as T is invertible, 2

$\Rightarrow \text{rank}\{[\tilde{B}\ \tilde{A}\tilde{B} \dots \tilde{A}^{n-1}\tilde{B}]\} = \text{rank}\{[B\ AB \dots A^{n-1}B]\}$

(c) $\text{rank}\{[B\ \ AB\ \dots\ A^{n-1}B]\}$

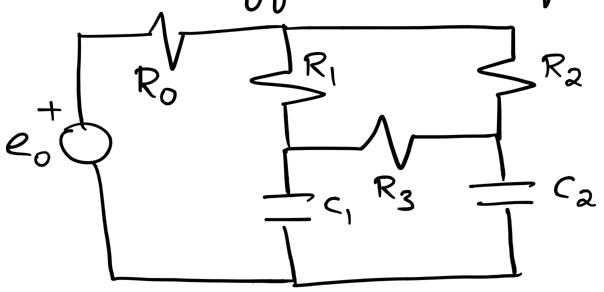
$= \text{rank}\{[\tilde{B}\ \ \tilde{A}\tilde{B}\ \dots\ \tilde{A}^{n-1}\tilde{B}]\}$

$= \text{rank}\left\{\begin{bmatrix} \tilde{B}_1 & \tilde{A}_{11}\tilde{B}_1 & \tilde{A}_{11}^2\tilde{B}_1 & \dots & \tilde{A}_{11}^{r-1}\tilde{B}_1 & \tilde{A}_{11}^{r}\tilde{B}_1 & \dots & \tilde{A}_{11}^{n-1}\tilde{B}_1 \\ 0 & 0 & 0 & & 0 & 0 & & 0 \end{bmatrix}\right\}$

$= r$ as there are r linearly independent rows. 2

Note, if $z = \begin{bmatrix} z_r \\ z_{n-r} \end{bmatrix} \begin{array}{l} \updownarrow r \\ \updownarrow n-r \end{array}$, then z_r states are controllable and z_{n-r} states are not controllable.

4. The differential equations for the bridge circuit



are
$$\dot{v}_1 = -\frac{1}{C_1}\left(\frac{1}{R_1}+\frac{1}{R_3}\right)v_1 + \frac{1}{C_1R_3}v_2 + \frac{1}{C_1R_1}e_0$$
$$\dot{v}_2 = \frac{1}{C_2R_3}v_1 - \frac{1}{C_2}\left(\frac{1}{R_2}+\frac{1}{R_3}\right)v_2 + \frac{1}{C_2R_2}e_0.$$

If the bridge is balanced $R_1C_1 = R_2C_2$, show that the state-space model with state $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and input e_0 is uncontrollable.

$\boxed{5 \text{ marks}}$

Ans. $A = \begin{bmatrix} -\frac{1}{C_1}\left(\frac{1}{R_1}+\frac{1}{R_3}\right) & \frac{1}{C_1R_3} \\ \frac{1}{C_2R_3} & -\frac{1}{C_2}\left(\frac{1}{R_2}+\frac{1}{R_3}\right) \end{bmatrix}$, $B = \begin{bmatrix} \frac{1}{C_1R_1} \\ \frac{1}{C_2R_2} \end{bmatrix}$

When bridge is balanced, $B = \frac{1}{C_1R_1}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (2)

$$W_c = [B \quad AB]$$
$$= \frac{1}{C_1R_1}\begin{bmatrix} 1 & -\frac{1}{C_1}\left(\frac{1}{R_1}+\frac{1}{R_3}\right) + \frac{1}{C_1R_3} \\ 1 & \frac{1}{C_2R_3} - \frac{1}{C_2}\left(\frac{1}{R_2}+\frac{1}{R_3}\right) \end{bmatrix}$$ (3)

$$= \frac{1}{C_1R_1}\begin{bmatrix} 1 & -\frac{1}{C_1R_1} \\ 1 & -\frac{1}{C_2R_2} \end{bmatrix}$$, whose rank $= 1$

$\Rightarrow$ Balanced bridge is uncontrollable.

5. Ideas of controllability and observability were introduced by R.E. Kalman in the mid 1950's as a way of explaining why a method of designing compensators for unstable systems by cancelling poles in the right half-plane by zeros in the right half-plane is doomed to fail even if the cancellation is perfect. Kalman showed that a perfect pole-zero cancellation would result in an unstable system with a stable transfer function. The transfer function, however, is of lower order than the system, and the unstable modes are either not capable of being affected by the input (uncontrollable) or not visible in the output (unobservable).

Consider the system below with input u and output y,

$$\dot{x}_1 = 2x_1 + 3x_2 + 2x_3 + x_4 + u$$

$$\dot{x}_2 = -2x_1 - 3x_2 - 2u$$

$$\boxed{8 \text{ marks}}$$

$$\dot{x}_3 = -2x_1 - 2x_2 - 4x_3 + 2u$$

$$\dot{x}_4 = -2x_2 - 2x_3 - 2x_5 - 5x_4 - u$$

$$y = 7x_1 + 6x_2 + 4x_3 + 2x_4.$$

The transfer function from $u \to y$ is $H(s) = \dfrac{1}{s+1}$

(a) Obtain a state-space model with variables $x = [x_1 \quad x_2 \quad x_3 \quad x_4]^T$.

(b) Transform the co-ordinates into $z = Tx$,

$$T = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

(c) Using the transformed model, show that the system is neither controllable nor observable.

(d) Explicitly write down the differential equations of the transformed model $z = [z_1 \ z_2 \ z_3 \ z_4]^T$ and identify the variables that are affected by the input and the variables that are visible in the output.

Ans. (a)

$$A = \begin{bmatrix} 2 & 3 & 2 & 1 \\ -2 & -3 & 0 & 0 \\ -2 & -2 & -4 & 0 \\ -2 & -2 & -2 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -2 \\ 2 \\ -1 \end{bmatrix}, \quad C = [7 \ 6 \ 4 \ 2]$$

1

(b) $\dot{z} = T\dot{x} = TAT^{-1}z + TBu$

$y = CT^{-1}z$

$$\underbrace{TAT^{-1}}_{\tilde{A}} = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 2 & 1 \\ -2 & -3 & 0 & 0 \\ -2 & -2 & -4 & 0 \\ -2 & -2 & -2 & -5 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 & 0 & 0 \\ 1 & -4 & 3 & 0 \\ 0 & 2 & -6 & 4 \\ 0 & 0 & 3 & -8 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

$$\underbrace{TB}_{\tilde{B}} = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\underbrace{CT^{-1}}_{\tilde{C}} = [7 \ 6 \ 4 \ 2] \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} = [1 \ 1 \ 0 \ 0]$$

2

(c) $$[\tilde{B} \ \ \tilde{A}\tilde{B} \ \ \tilde{A}^2\tilde{B} \ \ \tilde{A}^3\tilde{B}] = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & -3 & 9 & -27 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rightarrow$ rank 2

$\Rightarrow$ not controllable

$$\begin{bmatrix} \tilde{C} \\ \tilde{C}\tilde{A} \\ \tilde{C}\tilde{A}^2 \\ \tilde{C}\tilde{A}^3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ -1 & -8 & 0 & 0 \end{bmatrix} \rightarrow \text{rank } 2$$

$\Rightarrow$ not observable.

2

(d)

$$\left.\begin{aligned} \dot{z}_1 &= -z_1 + u \\ \dot{z}_2 &= -2z_2 \\ \dot{z}_3 &= -3z_3 + u \\ \dot{z}_4 &= -4z_4 \\ y &= z_1 + z_2 \end{aligned}\right\} \Rightarrow$$

z_1, z_3 affected by input

(z_2, z_4 not affected by input)

z_1, z_2 visible at output

(z_3, z_4 not visible at output)

3