ELL 333
08.08.2019

Last time: Solving $\dot{x} = Ax + Bu$

$$x(t) = e^{At} c(t)$$

searching for particular solution from homogeneous solution

$$\Rightarrow \dot{x} = \frac{d}{dt} e^{At} \cdot c(t)$$

$$\Rightarrow e^{At} \frac{d}{dt} c(t) = Bu$$

Next?
To separate $c(t)$, we multiply by $(e^{At})^{-1}$, assuming it exists.

$$\Rightarrow \frac{d}{dt} c(t) = (e^{At})^{-1} Bu$$

$$\Rightarrow c(t) = c(0) + \int_0^t (e^{A\tau})^{-1} Bu \, d\tau$$

Invertibility of e^{At}? $\triangleq I + tA + \frac{t^2}{2!}A^2 + \dots$

- Is it invertible? yes
- What is inverse? e^{-At}

1 answer: if eigenvalues unique, then yes

$$e^{At} = V e^{Dt} V^{-1} \quad \Rightarrow (e^{At})^{-1} = V (e^{Dt})^{-1} V^{-1}$$

$$(Ve^{Dt}V^{-1})^{-1} = (V^{-1})^{-1}(Ve^{Dt})^{-1}$$
$$= V(e^{Dt})^{-1}V^{-1}$$

$$e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & \\ & \ddots & \\ & & e^{\lambda_2 t} \end{bmatrix}, \quad (e^{Dt})^{-1} = \begin{bmatrix} e^{-\lambda_1 t} & \\ & \ddots & \\ & & e^{-\lambda_2 t} \end{bmatrix}$$
$$= e^{-Dt}$$

- $(e^{Dt})^{-1} = e^{-Dt}$
- if A has distinct eigenvalues, e^{At} is invertible.

Want to show $(e^{At})^{-1} = e^{-At}$ for all square matrices A.

1. $e^{A(t_1+t_2)} = e^{At_1} \cdot e^{At_2}$

2. Use $t_1 = +t$, $t_2 = -t$ and $e^{A0} = I$
$\Rightarrow \quad I = e^{At} \cdot e^{-At}$
Illy, using $t_1 = -t$, $t_2 = +t$
$\Rightarrow \quad I = e^{-At} \cdot e^{At}$
$\Rightarrow \quad (e^{At})^{-1} = e^{-At}$

2 should be straightforward.
For 1, use definition,

$$e^{A(t_1+t_2)} = I + (t_1+t_2)A + \frac{(t_1+t_2)^2}{2!}A^2 + \dots$$

$$e^{At_1} \cdot e^{At_2} = \left[I + t_1 A + \frac{t_1^2}{2!}A^2 + \dots\right]\left[I + t_2 A + \frac{t_2^2}{2!}A^2 + \dots\right]$$

$$= I + [t_1 A + t_2 A] + \left[\frac{t_1^2}{2!}A^2 + t_1 t_2 A^2 + \frac{t_2^2}{2!}A^2\right] + \dots$$

$$= I + (t_1+t_2)A + \frac{(t_1+t_2)^2}{2!}A^2 + \dots$$

$$= e^{A(t_1+t_2)}$$

$$\therefore \left(e^{At}\right)^{-1} = e^{-At}$$

We want to use it in

$$c(t) = c(0) + \int_0^t \left(e^{Az}\right)^{-1} B\, u(z)\, dz$$

$$\Rightarrow c(t) = c(0) + \int_0^t e^{-Az} B u(z)\, dz$$

$\therefore$ Overall solution of $\dot{x} = Ax + Bu$ is

$$x(t) = e^{At} x(0) + e^{At}\left[c(0) + \int_0^t e^{-Az} B u(z)\, dz\right]$$

To determine $c(0)$ put $t=0$

$$x(0) = \underbrace{e^{A0}}_{I} x(0) + \underbrace{e^{A0}}_{I}[c(0) + 0]$$

$$\Rightarrow \quad c(0) = 0$$

$\therefore$ The solution to $\dot{x} = Ax + Bu$ with initial condition $x(0)$ is

$$x(t) = e^{At} x(0) + e^{At}\int_0^t e^{-A\tau} Bu(\tau)\, d\tau$$

Maybe familiar in Laplace domain

$$\dot{x} = Ax + Bu$$

$$sX(s) - x(0) = AX(s) + BU(s)$$

$$\Rightarrow (sI - A)X(s) = x(0) + BU(s)$$

$$\Rightarrow X(s) = (sI-A)^{-1}x(0) + (sI-A)^{-1}BU(s)$$

$$x(t) = e^{At}x(0) + \underbrace{\int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau}_{\text{"similar" to convolution integral}}$$

Can be shown that it is same as

Quiz 2

Name

Entry Number

Q. a) Consider $\dot{z} = -z + u$, $z(0) = 0$

Can 'u' be used to reach $z(t) = 1$ at $t = 1$? What u can be used?

b) What do you think it means when we say that we can control a system $\dot{x} = Ax + Bu$?