ELL333

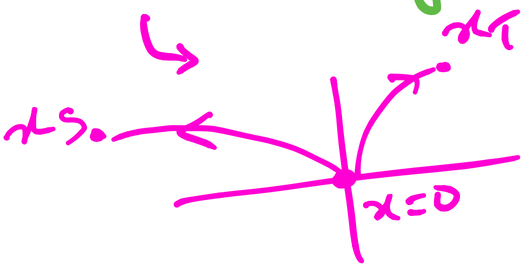
09.08.2019

(defn.) Controllability

A system $\dot{x} = Ax + Bu$ is said to be controllable if it is possible to have an input $u(t)$, $0 \le t \le T$, that can drive the system from $x(0) = 0$ (⊛) ($x(t)$ at $t=0$) to any $x(T)$ ($x(t)$ at $t=T$) in a finite time T.

⊛ instead of $x(0)$, we can consider $x(t_1)$ → because of time invariance

⊛ instead of $x(0)=0$, we can have any $x(0)$.



Because we can go from

- $x(0)=0$ to $x(0)=x_S$ if controllable
- then reverse time $(t \leftrightarrow -t)$ and go from x_S to $x(0)=0$
- and finally to x_T

In general, many definitions of controllability for different contexts are there. This one is the simplest for us.

$$x(t) = e^{At}x(0) + \underbrace{\int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau}$$

There is a controllability grammian

$$P(T) = \int_0^T \underbrace{\underset{n\times n}{e^{A(T-t)}}\ \underset{n\times m}{B}\ \underset{m\times n}{B'}\ \underset{n\times n}{e^{A'(T-t)}}}_{n\times n}\, dt$$

Thm. (Kalman) Controllable $\Longleftrightarrow$ $P(T)$ is non-singular

Proof.

$(\Longleftarrow)$ Suppose $P(T)$ is non-singular

We want to show that we can find a $u(t)$, $0 \le t \le T$, to reach any $x(T)$ from $x(0) = 0$.

$$u(t) = \underbrace{\underset{m\times n}{B'}\ \underset{n\times n}{e^{A'(T-t)}} \cdot \underset{n\times n}{P(T)^{-1}} \cdot \underset{n\times 1}{x(T)}}_{m\times 1}$$

Chosen u like this

used the solution

Solution for $x(0) = 0$ is

$$x(t) = \int_0^t e^{A(t-\tau)} B\, u(\tau)\, d\tau$$

$$= \int_0^t e^{A(t-\tau)} B \left(B' e^{A'(T-\tau)} P^{-1}(T).x(T)\right) d\tau$$

$$= \int_0^t e^{A(t-\tau)} B B' e^{A'(T-\tau)} d\tau \, . \, P^{-1}(T).x(T)$$

at $t = T$,

$$x(T) = \overbrace{\int_0^T e^{A(T-\tau)} B B' e^{A'(T-\tau)} d\tau}^{P(T)} \, . \, P^{-1}(T).x(T)$$

$$= x(T)$$

$\therefore$ it reaches any $x(T) \Rightarrow$ controllable.

($\Rightarrow$) controllability $\Rightarrow$ non-singularity of $P(T)$

Given that system is controllable.

Suppose $P(T)$ is singular.

(do we reach a contradiction?)

$\Rightarrow$ there is a non-zero vector v $(\neq 0)$ such that

$$P v = 0$$

$$v' P v = 0$$

$$\Rightarrow v'\left[\int_0^T e^{A(T-t)} B B' e^{A'(T-t)}\, dt\right] v = 0$$

$$\Rightarrow \int_0^T \underbrace{\left(v' e^{A(T-t)} B\right)}_{1 \times m} \underbrace{\left(B' e^{A'(T-t)} v\right)}_{m \times 1} dt = 0$$

Denote $z = B' e^{A'(T-t)} v$

$$\Rightarrow \int_0^T z' z \, dt = 0$$

if $z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{bmatrix} \Rightarrow \int_0^T (z_1^2 + z_2^2 + \dots z_m^2)\, dt = 0$

as integral of non-negative quantity over finite interval is zero, it means each term is zero.

$$\Rightarrow z = 0$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow B' e^{A'(T-t)} v = 0$$

At macro-level, want to show not controllable. Does this give a clue to what states cannot be reached?

Solution

$$\hookrightarrow x(t) = \int_0^t e^{A(t-\tau)} B u(\tau) d\tau$$