ELL333

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Solution

$$x(t) = e^{At}x(0) + \underbrace{\int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau}$$

There is a controllability grammian

$$P(T) = \int_0^T e^{A(T-t)} B B' e^{A'(T-t)} \, dt$$

Thm (Kalman): Controllable ($\dot{x} = Ax + Bu$) $\iff$ $P(T)$ is non-singular

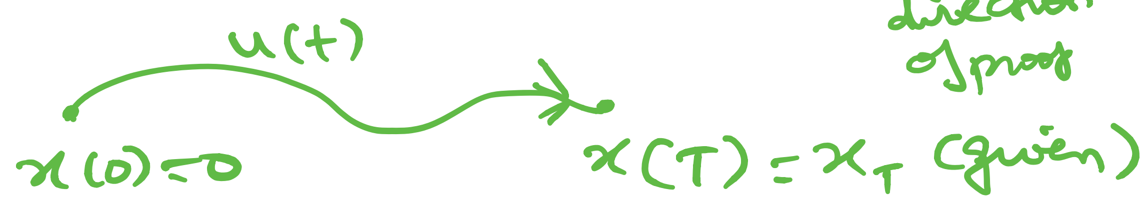
Proof.

($\Longleftarrow$) Suppose $P(T)$ is non-singular

We want to show that we can find a $u(t)$, $0 \le t \le T$, to reach any $x(T)$ from $x(0) = 0$.

$$u(t) = B' e^{A'(T-t)} \cdot P(T)^{-1} \cdot x(T)$$

assumed in this direction of proof



Solution for $x(0) = 0$ is

$$x(t) = \int_0^t e^{A(t-\tau)} B u(\tau) \, d\tau$$

$$= \int_0^t e^{A(t-\tau)} B \underbrace{\left(B' e^{A'(T-\tau)} P^{-1}(T) . x(T)\right)}_{u(\tau)} d\tau$$

$$= \int_0^t e^{A(t-\tau)} B B' e^{A'(T-\tau)} d\tau . P^{-1}(T) . x(T)$$

at $t = T$,

$$x(T) = \overbrace{\int_0^T e^{A(T-\tau)} B B' e^{A'(T-\tau)} d\tau}^{P(T)} . P^{-1}(T) . x(T)$$

$$= x(T)$$

$\therefore$ it reaches any $x(T)$ $\Rightarrow$ controllable.

not any input, we are showing for any final state.

($\Rightarrow$) controllability $\Rightarrow$ non-singularity of $P(T)$

Given that system is controllable.

Suppose $P(T)$ is singular.

(do we reach a contradiction?)

$\Rightarrow$ there is a non-zero vector v ($\neq 0$) such that

$$P v = 0$$

$$v' P v = 0$$

$$\Rightarrow \quad v' \left[\int_0^T e^{A(T-t)} B B' e^{A'(T-t)} dt \right] v = 0$$

$$\Rightarrow \int_0^T \left(v' e^{A(T-t)} B \right) \left(B' e^{A'(T-t)} v \right) dt = 0$$

Denote $z = B' e^{A'(T-t)} v$

$$\Rightarrow \quad \int_0^T z' z \, dt = 0$$

if $z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{bmatrix} \Rightarrow \int_0^T (z_1^2 + z_2^2 + \dots z_m^2) dt = 0$

as integral of non-negative quantity over finite interval is zero, it means each term is zero.

$\Rightarrow \quad z = 0 \quad \Rightarrow z' = 0$ also $\rightarrow v' e^{A(T-t)} B = 0$

$$\Rightarrow \quad B' e^{A'(T-t)} v = 0$$

At macro-level, want to show not controllable. Does this give a clue to what states cannot be reached?

Solution $\rightarrow x(t) = \int_0^t e^{A(t-\tau)} B u(\tau) d\tau$

Can we reach the state $x(T) = v$?

If we can, then

$$v = \int_0^T e^{A(T-z)} B u(z)\, dz$$

Multiply by v' on both sides,

$$\underbrace{v' v}_{\substack{\text{non-zero} \\ \text{as } v \neq 0 \\ \Rightarrow \text{LHS} \neq 0}} = \int_0^T \underbrace{v' e^{A(T-z)} B}_{\substack{\text{zero} \\ \text{as } v' e^{A(T-z)} B = 0 \\ \Rightarrow \text{RHS} = 0}} u(z)\, dz$$

This is a contradiction

$\therefore$ if controllable $\Rightarrow$ grammian is non-singular.

Compute this for $B = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $A = \begin{bmatrix} -1 & & & \\ & -2 & & \\ & & -3 & \\ & & & -4 \end{bmatrix}$

$$P(T) = \int_0^T e^{A(T-t)} B B' e^{A'(T-t)}\, dt$$

$$e^{A(T-t)} = \begin{bmatrix} e^{-(T-t)} & & & \\ & e^{-2(T-t)} & & \\ & & e^{-3(T-t)} & \\ & & & e^{-4(T-t)} \end{bmatrix}$$

$$= e^{A'(T-t)}$$

$\forall$ rank 4

$$BB' = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \text{rank } 1$$

$$\text{rank}\left\{ \underbrace{e^{A(T-t)} \underbrace{B \; B'}_{} e^{A'(T-t)}}_{} \right\} \leq 1$$

$$= e^{A(T-t)} \cdot \begin{bmatrix} x_1 & 0 & x_2 & 0 \\ 0 & 0 & 0 & 0 \\ x_3 & 0 & x_4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} x_5 & 0 & x_6 & 0 \\ 0 & 0 & 0 & 0 \\ x_7 & 0 & x_8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

'x's represents not a zero

$$P(T) = \int (\;) \, dt$$

$$\therefore \; P(T) \sim \begin{bmatrix} x_9 & 0 & x_{10} & 0 \\ 0 & 0 & 0 & 0 \\ x_{11} & 0 & x_{12} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ This is singular $\Rightarrow$ not controllable

Theorem

$P(T)$ is non-singular

$\Leftrightarrow$ rank $\{[B \;\; AB \;\; \dots \;\; A^{n-1}B]\} = n$

(n is order of system)

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \lambda-1 & 0 \\ 0 & \lambda \end{bmatrix}$$

$\Rightarrow \quad \lambda(\lambda-1) = 0 \Rightarrow \lambda = 0, 1$

($\Leftarrow$) Given rank $\{[B \;\; AB \;\; \dots \;\; A^{n-1}B]\} = n$

need to show that $P(T)$ is non-singular

Suppose $P(T)$ is singular

$\Rightarrow$ (from above proof)

there is a non-zero vector v such that $v' e^{A(T-t)} B = 0$

differentiate w.r.t 't' repeatedly,

$$v' e^{A(T-t)} . AB = 0$$

What is $\frac{d}{dt} e^{At} = A e^{At}$

or $e^{At} A$

or both ✓