Based on 16.08.2019 Lecture

Theorem

$P(T)$ is non-singular

$\Leftrightarrow$ rank $\{[B \ AB \ \dots \ A^{n-1}B]\} = n$

(n is order of system)

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \lambda-1 & 0 \\ 0 & \lambda \end{bmatrix}$$

$\therefore \quad \lambda(\lambda-1) = 0 \Rightarrow \lambda = 0, 1$

($\Leftarrow$) Given rank $\{[B \ AB \ \dots \ A^{n-1}B]\} = n$

need to show that $P(T)$ is non-singular.

Suppose $P(T)$ is singular

$\Rightarrow$ (from above proof)

there is a non-zero vector v such that $v' e^{A(T-t)} B = 0$

differentiate w.r.t 't' repeatedly,

$$v' e^{A(T-t)} \cdot AB = 0$$

$$v' e^{A(T-t)} \cdot A^2 B = 0$$

$$v' e^{A(T-t)} A^{n-1} B = 0$$

Combine in a matrix

$$v' e^{A(T-t)} [B \;\; AB \;\; \dots \;\; A^{n-1}B] = 0$$

As $v' e^{A(T-t)}$ is non-zero, this means $\text{rank}\{[B \;\; AB \dots A^{n-1}B]\} < n$

A contradiction.

$\therefore$ $P(T)$ has to be non-singular

($\Leftarrow$) Given $P(T)$ is non-singular

$$e^{A(T-t)} = I + A + A^2 \frac{t^2}{2!} + \dots$$

By Cayley-Hamilton theorem, A^n (and also A^k, $k > n$) can be expressed in powers of $A^0, A^1, \dots A^{n-1}$

$$\therefore e^{A(T-t)} = f_1(t,T) I + f_2(t,T) A + \dots f_n(t,T) A^{n-1}$$

$$\Rightarrow e^{A(T-t)} B = f_1(t,T) B + f_2(t,T) AB + \dots + f_n(t,T) A^{n-1} B$$

$$= \underbrace{[B \;\; AB \dots A^{n-1}B]}_{Q} \begin{bmatrix} f_1(t,T) I_{m\times m} \\ f_2(t,T) I_{m\times m} \\ \vdots \\ f_n(t,T) I_{m\times m} \end{bmatrix}$$

$$\Rightarrow P(T) = Q \int_0^T \begin{bmatrix} f_1(t,T) I_{m\times m} \\ f_2(t,T) I_{m\times m} \\ \vdots \\ f_n(t,T) I_{m\times m} \end{bmatrix} \times \left[\quad \right]' dt \; Q'$$

Suppose rank $\{Q\} < n$

$\Rightarrow$ rank of $P(T)$, which is a product of matrices involving Q, is also less than n.

$\#$ a contradiction.

$\therefore$ rank $\{Q\} = n$

Cayley - Hamilton Theorem (Bonus!)

From Friedland, Appendix

Every matrix satisfies its own characteristic polynomial.

Consider a matrix A of size $n \times n$ with characteristic polynomial

$$s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_n = 0$$

$$(sI - A)^{-1} = \frac{\operatorname{adj}(sI - A)}{\det(sI - A)}$$

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adj$(sI-A)$ is a matrix polynomial of degree $n-1$ i.e.

$$\text{adj}(sI-A) = E_1 s^{n-1} + E_2 s^{n-2} + \dots + E_n I,$$

where $E_1, E_2, \dots E_n$ are $n\times n$ matrices.

$$\therefore (sI-A)^{-1} \det(sI-A) = \text{adj}(sI-A)$$

multiply by $(sI-A)$

$$\Rightarrow I \det(sI-A) = (sI-A)\,\text{adj}(sI-A)$$

$$\text{LHS} = Is^n + a_1 I s^{n-1} + a_2 I s^{n-2} + \dots + a_n I$$

$$\text{RHS} = (sI-A)(E_1 s^{n-1} + E_2 s^{n-2} + \dots + E_n I)$$

$$\begin{aligned} = & E_1 s^n \\ & + (E_2 - AE_1) s^{n-1} \\ & + (E_3 - AE_2) s^{n-2} \\ & + \dots \\ & + (E_n - AE_{n-1}) s \\ & - AE_n \end{aligned}$$

Equating both sides,

$$E_1 = I$$

$$E_2 - AE_1 = a_1 I \Rightarrow E_2 = a_1 I + A$$

$$E_3 - AE_2 = a_2 I \Rightarrow E_3 = a_2 I + a_1 A + A^2$$

$$\dots$$

$$E_n - AE_{n-1} = a_{n-1} I$$

$$\Rightarrow E_n = a_{n-1} I + a_{n-2} A + \dots + A^{n-1}$$

$$-AE_n = a_n I$$

$$\Rightarrow a_n I + AE_n = 0$$

$$\Rightarrow a_n I + a_{n-1} A + a_{n-2} A^2 + \dots A^n = 0$$

which is what we want to show.