ELL 333

20.08.2019

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$\rightarrow 1, 1$ ✓

$\rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ✗

$$\dot{x} = Ax, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ eigenvector? anything

stable or unstable?

$\rightarrow$ eigenvalues > 0

$\dot{x}_1 = x_1$

$\dot{x}_2 = x_2$

Suppose we want to make it stable,

$$\dot{x} = Ax + Bu$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$u = -Kx = -\begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Can we choose k_1, k_2 so that overall system $\left\{ \begin{array}{l} \dot{x} = Ax + Bu \\ u = -Kx \end{array} \right\}$ is stable?

$$\dot{x} = Ax + B(-Kx)$$
$$= \underbrace{(A - BK)}\, x$$

How do its eigenvalues depend on k_1, k_2?

$$A - BK = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} k_1 & k_2 \\ k_1 & k_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-k_1 & -k_2 \\ -k_1 & 1-k_2 \end{bmatrix}$$

eigenvalues?

$$\det\left(sI - (A - BK)\right) = 0$$

$$\begin{vmatrix} s-1+k_1 & k_2 \\ +k_1 & s-1+k_2 \end{vmatrix} = 0$$

$$\Rightarrow (s-1+k_1)(s-1+k_2) - k_1 k_2 = 0$$

$$\Rightarrow (s-1)^2 + (k_1+k_2)(s-1) = 0$$

$$\Rightarrow (s-1)(s-1+k_1+k_2) = 0$$

$\therefore$ eigenvalues : $1, \; 1-k_1-k_2$

cannot be changed using k_1, k_2.

Is it surprising?

$\text{rank}\{[B \quad AB]\} = ?$

$$[B \quad AB] = \begin{bmatrix} | & | \\ | & | \end{bmatrix}$$

We want to say,

controllability condition

for single input,
if $\text{rank}\{[B \ AB \ \dots \ A^{n-1}B]\} = n$,
then we can place the eigenvalues of $A - BK$ arbitrarily through choice of K.

Companion Form

$$A = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix}$$

$$\text{char poly}(A) = s^2 + a_1 s + a_2$$

$$A = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\text{char poly}(A) = s^3 + a_1 s^2 + a_2 s + a_3 = 0$$

$$A = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\text{char poly}(A) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots a_n$$

Companion form is A and $B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

Advantage:

$$k = [k_1 \;\; k_2 \;\; \cdots \;\; k_n]$$

$$A - BK = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} [k_1 \;\; k_2 \cdots k_n]$$

$$= \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots 1 & 0 \end{bmatrix} - \begin{bmatrix} k_1 & k_2 & \cdots & k_n \\ & & 0 & \end{bmatrix}$$

$$= \begin{bmatrix} -(a_1+k_1) & -(a_2+k_2) & \cdots & -(a_n+k_n) \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots 1 & 0 \end{bmatrix}$$

char poly: $s^n + (a_1+k_1)s^{n-1} + \cdots + (a_n+k_n)$
which can have arbitrary roots depending on choice of K.

Check: $\text{rank}\{[B \;\; AB \; \ldots \; A^{n-1}B]\} = ?$

Quiz 3: $\dot{x} = Ax + Bu$,

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

a) What is $\text{rank}\{[B \;\; AB \;\; A^2B \;\; A^3B]\}$?

b) If $K = [k_1 \;\; k_2 \;\; k_3 \;\; k_4]$, can all eigenvalues of $A - BK$ be arbitrarily placed by choosing k_1, k_2, k_3, k_4?