

ELL333  
22.08.2019

Mingqi  
25.08.2019  
4:5 PM  
LH108  
Bring Calculators

Quiz 3:  $\dot{x} = Ax + Bu$ ,

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

a) What is  $\text{rank}\{[B \ AB \ A^2B \ A^3B]\}$ ?

b) If  $K = [k_1 \ k_2 \ k_3 \ k_4]$ , can all eigenvalues of  $A - BK$  be arbitrarily placed by choosing  $k_1, k_2, k_3, k_4$ ?

②

a)  $[B \ AB \ A^2B \ A^3B]$

$$= \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & -3 & 9 & -27 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{rank} = 2$$

no

b)  $A - BK = A - \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} [k_1 \ k_2 \ k_3 \ k_4]$

$$= A - \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \\ 0 & 0 & 0 & 0 \\ k_1 & k_2 & k_3 & k_4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1-k_1 & -k_2-k_3-k_4 \\ 0 & -2 & 0 & 0 \\ -k_1 & -k_2-3k_3-k_4 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

$$sI - (A - BK) = \begin{bmatrix} s+1+k_1 & k_2 & k_3 & k_4 \\ 0 & s+2 & 0 & 0 \\ k_1 & k_2 & s+3+k_3 & k_4 \\ 0 & 0 & 0 & s+4 \end{bmatrix}$$

$$\det(sI - (A - BK)) = (s+2) \begin{vmatrix} s+1+k_1 & k_3 & k_4 \\ k_1 & s+3+k_3 & k_4 \\ 0 & 0 & s+4 \end{vmatrix}$$

Single input case  
 $n$  eigenvalues to be placed,  
 $n$  unknowns to be obtained

$$K = [k_1, k_2, \dots, k_n]$$

Can this be done?

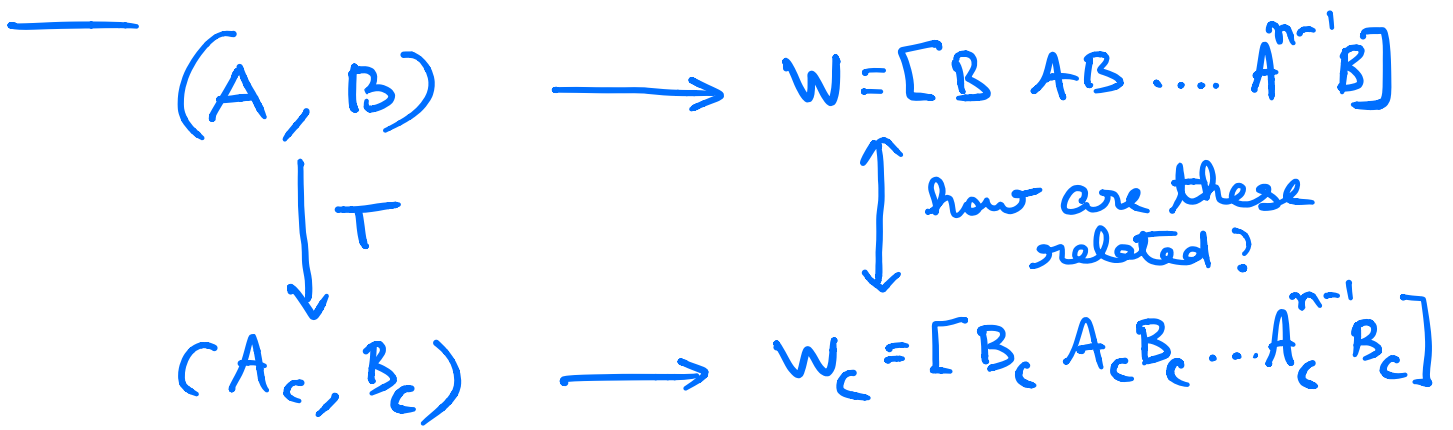
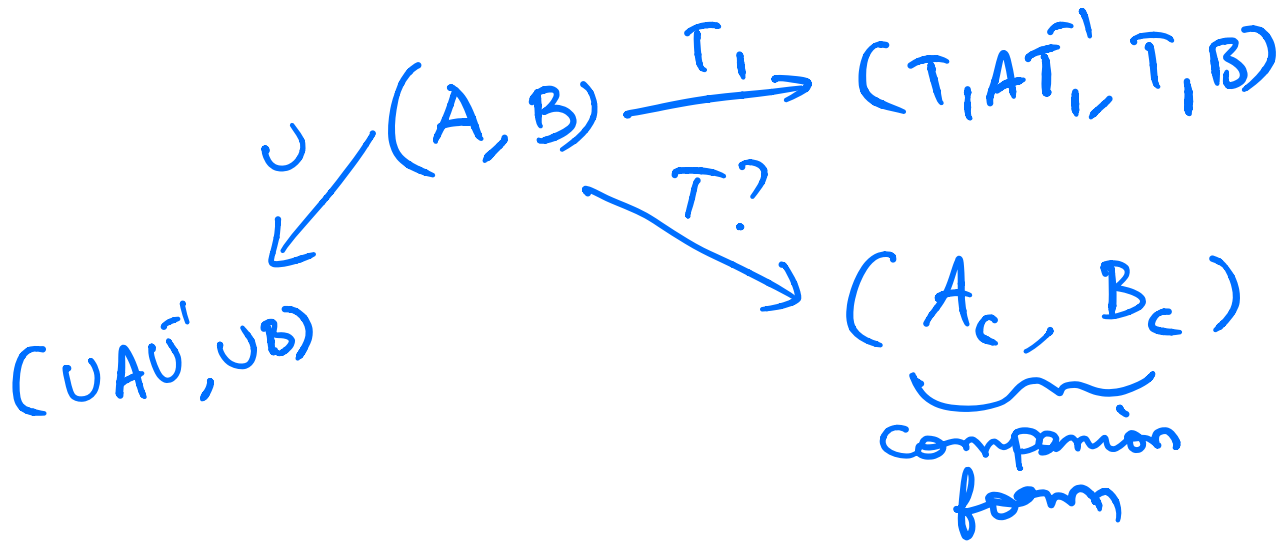
For companion form

$$A_c = \begin{bmatrix} -a_1 & -a_2 & \dots & -a_n \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}, B_c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

it can be done.

Question: Can it be done for a general system  $\dot{x} = Ax + Bu$ ?  
 (single-input)

Can a system  $\dot{x} = Ax + Bu$  be transformed into the companion form?



$$A_c = TAT^{-1}, B_c = TB$$

$$\begin{aligned}
 W_c &= [B_c \ A_c B_c \ \dots \ A_c^{n-1} B_c] \\
 &= [TB \ TAT^{-1}TB \ \dots \ \underbrace{(TAT^{-1}) \dots (TAT^{-1})}_{(n-1) \text{ times}} TB] \\
 &= [TB \ TAB \ \dots \ T A^{n-1} B] \\
 &= T [B \ AB \ \dots \ A^{n-1} B] \\
 &= TW
 \end{aligned}$$

$$W_c = T W$$

Given  $A, B,$

is  $W$  known? Yes

is  $W_c$  known? Yes

$$\rightarrow B_c = \begin{bmatrix} 1 \\ 0 \\ \vdots \end{bmatrix} \quad \text{Yes}$$

$$\rightarrow A_c = \begin{bmatrix} \text{char poly} \\ 1, 0 \end{bmatrix} \quad \text{Yes}$$



If  $W$  is invertible, then  $T = W_c W^{-1}$

$W$  is invertible  $\equiv \text{rank} [B \ AB \ \dots \ A^{n-1} B] = n$

•  $(A_c, B_c) \rightarrow$  can arbitrarily place eigenvalues  
 $\dot{z} = A_c z + B_c u$

•  $(A, B) \rightarrow$  can we do the same?  
 $\dot{x} = Ax + Bu$

• If  $W = [B \ AB \ \dots \ A^{n-1} B]$  has rank  $n$ ,  
 $x \rightarrow z$

$$\begin{aligned} z &= T x \\ A_c &= T A T^{-1} \\ B_c &= T B \end{aligned}$$

then  $T = W_c W^{-1}$  transforms  $(A, B) \rightarrow (A_c, B_c)$   
 where eigenvalues can be arbitrarily placed.

of  $A_c - B_c K$

$$\dot{z} = (A_c - B_c K) z \rightarrow [k_1, k_2, \dots, k_n]$$

$$\begin{aligned}
 x &= T^{-1} z \quad \Rightarrow \quad \dot{x} = T^{-1} \dot{z} \\
 &= T^{-1} (A_c - B_c K) z \\
 &= T^{-1} (A_c - B_c K) T x \\
 &= (T^{-1} A_c T - T^{-1} B_c \cdot K T) x \\
 &= (A - B \cdot (K T)) x
 \end{aligned}$$

$\therefore$  If  $\text{rank}\{[B \ AB \ \dots \ A^{n-1}B]\} = n$ , then eigenvalues of  $\dot{x} = Ax + Bu$ ,  $u = -Kx$ , may be arbitrarily placed through choice of  $K$ .

Question: Can a given system  $\dot{x} = Ax + Bu$  be transformed into this?

How does transformation help?

$$z = Tx, \quad T^{-1} \text{ exist}$$

$$\dot{z} = T \dot{x} = TA x + TB u$$

$$\Rightarrow \dot{z} = TAT^{-1} z + TB u$$

$A$  &  $TAT^{-1}$  have same characteristic polynomial and eigenvalues