ELL333
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Minor 1
25.08.2019
4-5 PM
LH108
Bring calculators

Quiz 3: $\dot{x} = Ax + Bu$,

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

a) What is rank$\{[B \; AB \; A^2B \; A^3B]\}$?

b) If $K = [k_1 \; k_2 \; k_3 \; k_4]$, can all eigenvalues of $A - BK$ be arbitrarily placed by choosing k_1, k_2, k_3, k_4?

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a) $[B \; AB \; A^2B \; A^3B]$

$$= \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & -3 & 9 & -27 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

rank = 2

no

b) $A - BK = A - \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} [k_1 \; k_2 \; k_3 \; k_4]$

$$= A - \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \\ 0 & 0 & 0 & 0 \\ k_1 & k_2 & k_3 & k_4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1-k_1 & -k_2 & -k_3 & -k_4 \\ 0 & -2 & 0 & 0 \\ -k_1 & -k_2 & -3-k_3 & -k_4 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

$$sI-(A-BK) = \begin{bmatrix} s+1+k_1 & k_2 & k_3 & k_4 \\ 0 & s+2 & 0 & 0 \\ k_1 & k_2 & s+3+k_3 & k_4 \\ 0 & 0 & 0 & s+4 \end{bmatrix}$$

$$\det(sI-(A-BK)) = (s+2) \begin{vmatrix} s+1+k_1 & k_3 & k_4 \\ k_1 & s+3+k_3 & k_4 \\ 0 & 0 & s+4 \end{vmatrix}$$

Single input case

$\underbrace{n \text{ eigenvalues to be placed, } n \text{ unknowns to be obtained}}_{K = [k_1 \; k_2 \; \dots \; k_n]}$

Can this be done?

For companion form

$$A_c = \begin{bmatrix} -a_1 & -a_2 & \dots & -a_n \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots 1 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

it can be done.

Question: Can it be done for a general (single-input) system $\dot{x} = Ax + Bu$?

Can a system $\dot{x} = Ax + Bu$ be transformed into the companion form?

$$
\begin{tikzcd}
 & (A, B) \arrow[r, "T_1"] \arrow[dl, "U"'] \arrow[dr, "T?"] & (T_1 A T_1^{-1}, T_1 B) \\
(UAU^{-1}, UB) & & \underbrace{(A_c, B_c)}_{\text{companion form}}
\end{tikzcd}
$$

$$
\begin{tikzcd}
(A, B) \arrow[r] \arrow[d, "T"] & W = [B \; AB \; \dots \; A^{n-1}B] \arrow[d, leftrightarrow, "\text{how are these related?}"] \\
(A_c, B_c) \arrow[r] & W_c = [B_c \; A_cB_c \dots A_c^{n-1}B_c]
\end{tikzcd}
$$

$A_c = TAT^{-1}, \; B_c = TB$

$$
\begin{aligned}
W_c &= [B_c \quad A_cB_c \quad \dots \quad A_c^{n-1}B_c] \\
&= [TB \quad TAT^{-1}TB \quad \dots \quad \underbrace{(TAT^{-1})\dots(TAT^{-1})}_{(n-1)\text{ times}}TB] \\
&= [TB \quad TAB \quad \dots \quad TA^{n-1}B] \\
&= T\,[B \quad AB \quad \dots \quad A^{n-1}B] \\
&= TW
\end{aligned}
$$

$$W_c = T W$$

Given A, B,

is W known? Yes

is W_c known? Yes

↳ $B_c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ Yes

→ $A_c = \begin{bmatrix} \text{char poly} \\ 1, 0 \end{bmatrix}$ Yes

If W is invertible, then $T = W_c W^{-1}$

W is invertible $\equiv$ rank $[B \; AB \ldots A^{n-1}B] = n$

- $(A_c, B_c) \longrightarrow$ can arbitrarily place eigenvalues

$\dot{z} = A_c z + B_c u$

- $(A, B) \longrightarrow$ can we do the same?

$\dot{x} = Ax + Bu$

$z = Tx$

$A_c = TAT^{-1}$

$B_c = TB$

- If $W = [B \; AB \ldots A^{n-1}B]$ has rank n,

$x \rightarrow z$

then $T = W_c W^{-1}$ transforms $(A,B) \rightarrow (A_c, B_c)$, where eigenvalues of $A_c - B_c K$ can be arbitrarily placed.

$\dot{z} = (A_c - B_c K) z \rightarrow [k_1, k_2 \ldots k_n]$

$$x = T^{-1}z \implies \dot{x} = T^{-1}\dot{z}$$
$$= T^{-1}(A_c - B_c K)z$$
$$= T^{-1}(A_c - B_c K)Tx$$
$$= (T^{-1}A_c T - T^{-1}B_c \cdot KT)x$$
$$= (A - B \cdot (KT))x$$

$\therefore$ If rank$\{[B \; AB \; \dots \; A^{n-1}B]\} = n$, then eigenvalues of $\dot{x} = Ax + Bu$, $u = -Kx$, may be arbitrarily placed through choice of K.

Question: Can a given system
$\dot{x} = Ax + Bu$ be transformed into this?

How does transformation help?

$z = Tx$, T^{-1} exist

$\dot{z} = T\dot{x} = TAx + TBu$

$\Rightarrow \dot{z} = TAT^{-1}z + TBu$

A & TAT^{-1} have same characteristic polynomial and eigenvalues