ELL 333
29.08.2019

- Example of eigenvalue assignment

$\dot{x} = Ax + Bu$ ← inverted pendulum

$A = \begin{bmatrix} 0 & 1 \\ \Omega^2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$(\Omega > 0)$

eigenvalues (A) are $\pm \Omega \Rightarrow$ Unstable

char poly (A) : $s^2 - \Omega^2 = 0$

Design objective : Use $u = -Kx$

$= -[k_1 \; k_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

to make it stable.

What eigenvalues we may want?

$\{0, -1\}$, $\{-10^6, -10^6\}$, $\{\alpha, \beta\}$

$\alpha < 0, \beta < 0$

$\therefore$ char poly $(A - BK) \equiv (s - \alpha)(s - \beta)$

$= s^2 - (\alpha + \beta)s + \alpha\beta$.

Direct method (because only two dimension)

$A - BK = \begin{bmatrix} 0 & 1 \\ \Omega^2 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} [k_1, \; k_2]$

$$= \begin{bmatrix} 0 & 1 \\ \Omega^2 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ \Omega^2 - k_1 & -k_2 \end{bmatrix}$$

char poly $(A-BK) = s^2 + k_2 s - (\Omega^2 - k_1)$

desired char poly $= s^2 - (\alpha+\beta)s + \alpha\beta$

$$k_1 - \Omega^2 = \alpha\beta$$

$$k_2 = -(\alpha+\beta)$$

$\therefore$ solution is $k_1 = \alpha\beta + \Omega^2 \leftarrow$

$k_2 = -(\alpha+\beta) \leftarrow$

$(0,-1) \rightarrow k_1 = \Omega^2, k_2 = 1$

$(-10^6, -10^6) \rightarrow k_1 = 10^{12} + \Omega^2, k_2 = 2\times 10^6$

$$u = -k_1 x_1 - k_2 x_2$$

- Large eigenvalues may need large control effort but give fast performance
- Optimize speed and control effort using a cost function $\int^T (u^2 + x^2)\,dt$

$u^2 \rightarrow u^T Q u \quad x^2 \rightarrow x^T R x$

LQR $\leftarrow$ ①

Linear Quadratic Regulator

Same design, but using companion form.

$$A = \begin{bmatrix} 0 & 1 \\ \Omega^2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- char poly $(A) = s^2 - \Omega^2$

Comp-anion

$$A_c = \begin{bmatrix} 0 & +\Omega^2 \\ 1 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

For this form $u = -Kx = -[k_1 \; k_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

What is k_1, k_2 ?

$$A_c - B_c K = \begin{bmatrix} -k_1 & \Omega^2 - k_2 \\ 1 & 0 \end{bmatrix}$$

$\dot{x}_c = A_c x_c + B_c u$

$u = -K x_c$

$$\text{char poly}(A_c - B_c K) = s^2 + k_1 s - (\Omega^2 - k_2)$$

same solution as before, $k_1 = -(\alpha + \beta)$

$k_2 - \Omega^2 = \alpha\beta$.

These k_1, k_2 are for

$u = -K x_c$

- $W = [B \quad AB]$

$W_c = [B_c \quad A_c B_c]$

If there is invertible transformation from $(A, B) \xrightarrow{T} (A_c, B_c)$, then

say $x_c = Tx \longrightarrow \dot{x} = Ax + Bu$

companion form variable

$$A_c = TAT^{-1}, \quad B_c = TB$$

$$W_c = TW$$

$$\Rightarrow \quad T = W_c W^{-1},$$ which exists only if W is invertible, which means system is controllable.

$$W = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad W_c = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \quad T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{(-1)} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\therefore \quad u = -Kx_c = -KTx$$

$$= -[k_1 \ \ k_2] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x$$

$$= -[k_2 \ \ k_1]\, x$$

Same solution as design in the direct method.

Some theoretical results for single input that may be used to motivate this design process.

1. If (A, B) is controllable then (TAT^{-1}, TB) is also controllable, where T is an invertible transformation.

$\dot{x} = Ax + Bu$
Controllable

$\xrightarrow[T^{-1} \text{ exists}]{z = Tx,}$

$\dot{z} = TAT^{-1} + TBu$
<u>also</u> controll-able

$$W = [B \quad AB \quad \dots \quad A^{n-1}B]$$

We know W is invertible & square (B is for single input) as (A, B) is controllable.

$$W_c = [TB \quad TAT^{-1}TB \dots (TAT^{-1})^{n-1} TB]$$
$$= T\,W$$

rank $(W) = n$ (as (A,B) controllable

rank $(T) = n$ (as T is invertible)

We want to show rank $(W_c) = n$?

- Suppose rank $(W_c) < n$
  then as $W = T^{-1} W_c$
  $\Rightarrow$ rank $(W) \leq$ rank $(W_c) < n$
  not possible because rank$(W) = n$.
- or $W_c^{-1} = W^{-1} T^{-1}$ works as inverse for W_c

2. (A, B) is controllable

$\iff$ there is an invertible transformation T that transforms (A, B) into the companion form.

i.e $\quad TAT^{-1} = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \\ 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots\ 1 & 0 \end{bmatrix}$

$$TB = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$(\Leftarrow)$ is companion form controllable ?

$$W_c = \begin{bmatrix} 1 & -a_1 & a_1^2 - a_2 & \\ 0 & 1 & -a_1 & \\ \vdots & \vdots & 1 & \\ 0 & 0 & \vdots\ 0 & 1 \end{bmatrix} \Rightarrow \text{rank}(W_c) = n$$